Contents

Dedication *v*

Preface *xv*

Acknowledgments *xvii*

Part I METHODS

1 The Basic Method *1*
1.1 The Probabilistic Method *1*
1.2 Graph Theory *3*
1.3 Combinatorics *7*
1.4 Combinatorial Number Theory *9*
1.5 Disjoint Pairs *10*
1.6 Independent Sets and List Coloring *11*
1.7 Exercises *14*

The Probabilistic Lens: The Erdős–Ko–Rado Theorem *16*

2 Linearity of Expectation *17*

2.1 Basics 17
2.2 Splitting Graphs 18
2.3 Two Quickies 20
2.4 Balancing Vectors 21
2.5 Unbalancing Lights 23
2.6 Without Coin Flips 24
2.7 Exercises 25

The Probabilistic Lens: Brégman's Theorem 27

3 Alterations 31
3.1 Ramsey Numbers 31
3.2 Independent Sets 33
3.3 Combinatorial Geometry 34
3.4 Packing 35
3.5 Greedy Coloring 36
3.6 Continuous Time 38
3.7 Exercises 41

The Probabilistic Lens: High Girth and High Chromatic Number 42

4 The Second Moment 45
4.1 Basics 45
4.2 Number Theory 46
4.3 More Basics 49
4.4 Random Graphs 51
4.5 Clique Number 55
4.6 Distinct Sums 56
4.7 The Rödl Nibble 58
4.8 Exercises 63

The Probabilistic Lens: Hamiltonian Paths 65

5 The Local Lemma 69
5.1 The Lemma 69
5.2 Property B and Multicolored Sets of Real Numbers 72
5.3 Lower Bounds for Ramsey Numbers 73
5.4 A Geometric Result 75

5.5 The Linear Arboricity of Graphs *76*
5.6 Latin Transversals *80*
5.7 Moser's FIX-IT Algorithm *81*
5.8 Exercises *86*

The Probabilistic Lens: Directed Cycles *88*

6 Correlation Inequalities *91*
6.1 The Four Functions Theorem of Ahlswede and Daykin *92*
6.2 The FKG Inequality *95*
6.3 Monotone Properties *96*
6.4 Linear Extensions of Partially Ordered Sets *98*
6.5 Exercises *100*

The Probabilistic Lens: Turán's Theorem *101*

7 Martingales and Tight Concentration *103*
7.1 Definitions *103*
7.2 Large Deviations *105*
7.3 Chromatic Number *107*
7.4 Two General Settings *110*
7.5 Four Illustrations *113*
7.6 Talagrand's Inequality *116*
7.7 Applications of Talagrand's Inequality *119*
7.8 Kim–Vu Polynomial Concentration *121*
7.9 Exercises *122*

The Probabilistic Lens: Weierstrass Approximation Theorem *124*

8 The Poisson Paradigm *127*
8.1 The Janson Inequalities *127*
8.2 The Proofs *129*
8.3 Brun's Sieve *132*
8.4 Large Deviations *135*
8.5 Counting Extensions *137*
8.6 Counting Representations *138*
8.7 Further Inequalities *141*

8.8 Exercises *143*

The Probabilistic Lens: Local Coloring *144*

9 Quasirandomness *147*
9.1 The Quadratic Residue Tournaments *148*
9.2 Eigenvalues and Expanders *151*
9.3 Quasirandom Graphs *157*
9.4 Szemerédi's Regularity Lemma *165*
9.5 Graphons *169*
9.6 Exercises *172*

The Probabilistic Lens: Random Walks *173*

Part II TOPICS

10 Random Graphs *177*
10.1 Subgraphs *178*
10.2 Clique Number *180*
10.3 Chromatic Number *182*
10.4 Zero-One Laws *183*
10.5 Exercises *191*

The Probabilistic Lens: Counting Subgraphs *192*

11 The Erdős–Rényi Phase Transition *195*
11.1 An Overview *196*
11.2 Three Processes *198*
11.3 The Galton–Watson Branching Process *199*
11.4 Analysis of the Poisson Branching Process *200*
11.5 The Graph Branching Model *202*
11.6 The Graph and Poisson Processes Compared *203*
11.7 The Parametrization Explained *205*
11.8 The Subcritical Regions *206*
11.9 The Supercritical Regimes *207*
11.10 The Critical Window *210*
11.11 Analogies to Classical Percolation Theory *212*

11.12 Exercises *216*

The Probabilistic Lens: Long paths in the supercritical regime *218*

12 Circuit Complexity *221*
12.1 Preliminaries *221*
12.2 Random Restrictions and Bounded-Depth Circuits *223*
12.3 More on Bounded-Depth Circuits *227*
12.4 Monotone Circuits *229*
12.5 Formulae *233*
12.6 Exercises *234*

The Probabilistic Lens: Maximal Antichains *235*

13 Discrepancy *237*
13.1 Basics *237*
13.2 Six Standard Deviations Suffice *239*
13.3 Linear and Hereditary Discrepancy *243*
13.4 Lower Bounds *245*
13.5 The Beck–Fiala Theorem *247*
13.6 Exercises *248*

The Probabilistic Lens: Unbalancing Lights *251*

14 Geometry *253*
14.1 The Greatest Angle Among Points in Euclidean Spaces *254*
14.2 Empty Triangles Determined by Points in the Plane *255*
14.3 Geometrical Realizations of Sign Matrices *257*
14.4 ϵ-Nets and VC-Dimensions of Range Spaces *259*
14.5 Dual Shatter Functions and Discrepancy *264*
14.6 Exercises *266*

The Probabilistic Lens: Efficient Packing *268*

15 Codes, Games and Entropy *271*
15.1 Codes *271*
15.2 Liar Game *274*
15.3 Tenure Game *276*

15.4 Balancing Vector Game 277
15.5 Nonadaptive Algorithms 279
15.6 Half Liar Game 280
15.7 Entropy 281
15.8 Exercises 287

The Probabilistic Lens: An Extremal Graph 289

16 Derandomization 291
16.1 The Method of Conditional Probabilities 291
16.2 d-Wise Independent Random Variables in Small Sample Spaces 295
16.3 Exercises 299

The Probabilistic Lens: Crossing Numbers, Incidences, Sums and Products 301

17 Graph Property Testing 305
17.1 Property Testing 305
17.2 Testing Colorability 306
17.3 Testing Triangle-Freeness 310
17.4 Characterizing the Testable Graph Properties 312
17.5 Exercises 314

The Probabilistic Lens: Turán Numbers and Dependent Random Choice 315

Appendix A: Bounding of Large Deviations 319
A.1 Chernoff Bounds 319
A.2 Lower Bounds 328
A.3 Exercises 332

The Probabilistic Lens: Triangle-Free Graphs Have Large Independence Numbers 333

Appendix B: Paul Erdős 335
B.1 Papers 335
B.2 Conjectures 337
B.3 On Erdős 338

B.4 Uncle Paul *339*

The Probabilistic Lens: The Rich Get Richer *342*

Appendix C: Hints to selected exercises *345*

References *351*

Author Index *370*

Subject Index *375*