Random Graphs G22.3033-007
Notes on Asymptotics

Let's start with the Taylor Series
\[
\ln(1 - \epsilon) = -\epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} \ldots
\]
valid for \(|\epsilon| < 1\) though we will only be interested in \(\epsilon\) small positive. This is too much information so we cut it down in a variety of ways:
\[
\ln(1 - \epsilon) \sim -\epsilon\text{ when } \epsilon = o(1) \quad (2)
\]
and with error term
\[
\ln(1 - \epsilon) = -\epsilon + O(\epsilon^2) \quad (3)
\]
Sometimes we need a more precise result
\[
\ln(1 - \epsilon) = -\epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3) \quad (4)
\]
While one could continue this sequence, these will suffice for this course.

Now let's examine the asymptotics of \(\binom{n}{k}\) when \(n, k \to \infty\). We write:
\[
\binom{n}{k} = \frac{(n)_k}{k!} \sim \frac{n^k e^k}{k^{k/2} k \pi} A \quad (5)
\]
where we set
\[
A := \frac{(n)_k}{n^k} = \prod_{i=0}^{k-1} (1 - \frac{i}{n}) \quad (6)
\]
So if we get \(A\) we get the binomial coefficient. It is more convenient to work with
\[
B := \ln A = \sum_{i=0}^{k-1} \ln(1 - \frac{i}{n}) \quad (7)
\]
For \(k = o(n)\) we have
\[
B \sim \sum_{i=0}^{k-1} -\frac{i}{n} \sim -\frac{k^2}{2n} \quad (8)
\]
and thus we can write
\[
A = e^{-\frac{k^2}{2n} (1 + o(1))} \quad (9)
\]
This does not give the full asymptotics of $A$ as the $1+o(1)$ is in the exponent. We go further as follows:

$$B = \sum_{i=0}^{k-1} -\frac{i}{n} + O(i^2 n^{-2}) = -\frac{k^2}{2n} + O(k^3 n^{-2}) \quad (10)$$

So if $k = o(n^{2/3})$, $B = -\frac{k^2}{2n} + o(1)$ and we have the asymptotic formula

$$A = e^{-\frac{k^2}{2n}} (1 + o(1)) \quad (11)$$

In particular:

If $k = o(n^{1/2})$ then $A \sim 1$ \quad (12)

If $k \sim cn^{1/2}$ then $A \sim e^{-\frac{c^2}{2}}$ \quad (13)

If $k = o(n^{3/4})$ we go to the next approximation:

$$B = \sum_{i=0}^{k-1} -\frac{i}{n} - \frac{i^2}{2n^2} + O(i^3 n^{-3}) = -\frac{k^2}{2n} - \frac{k^3}{6n^2} + O(k^4 n^{-3}) \quad (14)$$

and the error term is $o(1)$ so that we have the asymptotic formula

$$A = e^{-\frac{k^2}{2n}} e^{-\frac{k^3}{6n^2}} (1 + o(1)) \quad (15)$$

In particular (this case will come up a number times)

If $k \sim cn^{2/3}$ then $A \sim e^{-\frac{c^2}{2n}} e^{-\frac{c^3}{6n}}$ \quad (16)

BTW, the inequality

$$\ln(1 - \epsilon) < -\epsilon or, equivalently 1 - \epsilon < e^{-\epsilon} \quad (17)$$

is valid for all $\epsilon \in (0, 1)$ and can be pretty handy.