Distortion Bounds of Subdivision Models for $\boldsymbol{SO(3)}$

Chee Yap⋆ and Zhaoqi Zhang

[1] Courant Institute, New York University
New York, NY 10012, USA
yap@cs.nyu.edu
Homepage: https://cs.nyu.edu/yap

[2] Beijing Key Laboratory of Topological Statistics
& Applications for Complex Systems
Beijing Institute of Mathematical Sciences and Applications
Beijing 101408, China
zhangzhaoqi@bimsa.cn
Homepage: https://www.bimsa.cn/detail/ZhaoqiZhang.html

In Memoriam Stephen Cameron (1958–2019)

Abstract. In the subdivision approach to robot path planning, we need to subdivide the configuration space of a robot into nice cells to perform various computations. For a rigid spatial robot, this configuration space is $SE(3) = \mathbb{R}^3 \times SO(3)$. The subdivision of $\mathbb{R}^3$ is standard but so far, there are no global subdivision schemes for $SO(3)$. We recently introduced a representation for $SO(3)$ suitable for subdivision. This paper investigates the distortion of the natural metric on $SO(3)$ caused by our representation. The proper framework for this study lies in the Riemannian geometry of $SO(3)$, enabling us to obtain exact distortion bounds.

Keywords: subdivision path planning, subdivision atlas, robot path planning, cubic model of SO(3), distortion constant.

1 Introduction

Path planning is a fundamental task in robotics [4]. The problem may be formulated thus: Fix a robot R_0 in $\mathbb{R}^k$ $(k = 2, 3)$. For example, a rigid robot R_0 can be identified as a subset of $\mathbb{R}^k$ (typically a disc or convex polygon). Given (α, β, Ω) where α, β are the start and goal configurations of R_0, the task is to either find a path from α to β avoiding the obstacle set $\Omega \subseteq \mathbb{R}^k$, or output NO-PATH. Such an algorithm is called a **planner** for R_0. This problem originated in AI as the FINDPATH problem. In the 1980s, path planning began to be studied algorithmically, from an intrinsic geometric perspective. Schwartz and Sharir showed

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that algebraic path planning can be solved exactly by a reduction to cylindrical algebraic decomposition. Yap (1987) described two "universal methods" for constructing such paths: cell-decomposition and retraction. These exact methods are largely of theoretical interest because they require exact computation with algebraic numbers. Since no physical robot is exact and maps of the world even less so, we need numerical approximations. But we lack a systematic way to "approximately implement exact algorithms" (this difficulty is not specific to path planning). After the 1990s the exact approach is largely eclipsed by the sampling approach (combined with randomness) such as PRM, RRT and many variants etc. These proved to be practical, widely applicable and easy to implement [4]. But it has a well-known bane called the "narrow passage" problem This bottleneck (sic) is actually symptomatic of a deeper problem, namely, sampling algorithms do not know how to halting when there is no path [8].

In [8,9], we revisited the subdivision approach by introducing the "Soft Subdivision Search" (SSS) framework to address two foundational issues: (1) To avoid the underlying cause of the above halting problem, we introduce the notion of **resolution-exactness**. (2) To exploit resolution-exactness, we need the notion of **soft-predicates**. In a series of papers with implementations [8,10], we showed that SSS framework is practical. The guarantee of resolution-exactness is much stronger than any guarantees of sampling algorithms. Despite such strong guarantees, SSS planners outperform or match the state-of-art sampling algorithms for various robots with up to 6 degrees of freedom (DOF). Our last paper [10] reached a well-known milestone, achieving the first rigorous, complete and implementable path planner for a rigid spatial robot with 6 DOF.

Central to the design and implementation of the 6-DOF planner [10] is a representation of $SO(3)$ that supports subdivision. As a 3-dimensional space, $SO(3)$ can be *locally* represented by three real parameters. E.g., using Euler angles (α, β, γ) which range over the box $B_0 = [-\pi, \pi] \times [-\pi/2, \pi/2] \times [-\pi, \pi] \subseteq \mathbb{R}^3$. Such parametrizations have well-known singularities ($\beta = 0$) and a wrong global topology. For example for there is By viewing $SO(3)$ as quaternions, we can

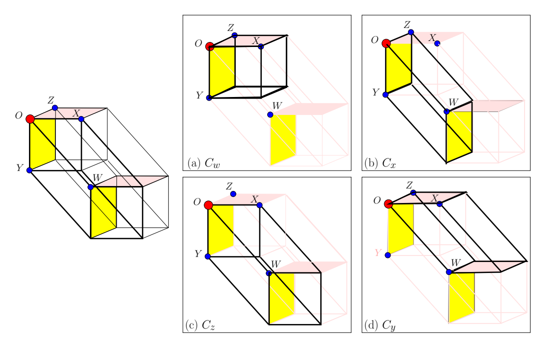

Fig. 1. The Cubic model $\widehat{SO}_3$ for $SO(3)$ (taken from [9])

represent it by the boundary $\partial[-1,1]^4$ of the 4-dimensional cube $[-1,1]^4$, after identification of opposite pairs of faces. After the identification, we have 4 copies of the standard 3-cube $C = [-1,1]^3 \subseteq \mathbb{R}^3$ which are shown as C_w, C_x, C_y, C_z in Figure 1. We can now do subdivision on these cubes. This "cubic model"[3] of $SO(3)$ was known to Canny [1, p. 36], and to Nowakiewicz [6]. To our knowledge, this model has never been systematically developed before. New data structures and algorithms for this representation are needed [10]. This paper addresses a mathematical question about this representation.

Brief Literature Overview. Besides the above introduction to path planning, there are many surveys [11]. In this paper, we study $SO(3)$ as a metric space. Since $3D$ rotations arise in applications such as computer vision and graphics, many $SO(3)$ metrics are known. Huynh [3] listed six of these metrics Φ_i $(i = 1, \ldots, 6)$. We will focus on Φ_6, simply calling it the **natural distance** for $SO(3)$ because it has all the desirable properties and respects the Lie group structure of $SO(3)$. Basically, $\Phi_6(R_1, R_2)$ is the angle of the rotation $R_1^{-1}R_2$ about its rotation-axis.

More complete references and any missing proofs in this paper may be found in the arXiv version of this paper [11].

2 Distortion in Cubic Models for S^n

We call $\widehat{S^n} := \partial([-1,1]^{n+1})$ the **cubic model** of S^n and consider the homeomorphism $\mu_n : \widehat{S^n} \to S^n$ where

$$\mu_n(q) := q/\|q\|_2 \quad \text{and} \quad \mu_n^{-1}(q) := q/\|q\|_\infty \tag{1}$$

and $\|q\|_p$ denotes the p-norm. Viewing S^n and $\widehat{S^n}$ as metric spaces with the induced metric [7, p. 3], the **cubic representation** μ_n introduces a distortion in the distance function of S^n. We want to bound this distortion.

In general, if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is continuous, we define the **distortion range** of f to be the closure of the set

$$D_f := \left\{ \frac{d_Y(f(p), f(q))}{d_X(p,q)} : p, q \in X, p \neq q \right\}. \tag{2}$$

If the distortion range is $[a, b]$, then the **distortion constant** of f is

$$C_0(f) := \max\{b, 1/a\}. \tag{3}$$

Note that $C_0(f) \geqslant 1$ is the largest expansion or contraction factor produced by the map f. In [9], $C_0(f)$ was introduced as the subdivision atlas constant. If $C_0(f) = 1$, then f is just an isometry. In [2], we showed[4] that the map μ_2

[3] We are indebted to the late Stephen Cameron who first brought it to our attention (June 2018). This paper is dedicated to his memory.

[4] We originally claimed that the range is $[1/\sqrt{3}, 1]$; Zhaoqi pointed out that the correct range is $[1/3, 1]$.

has distortion range $[\frac{1}{3}, 1]$. The proof for μ_2 used elementary geometry which is not easily generalized to μ_3. In this paper, we provide the proper mathematical framework for a generalization to any μ_n. This paper will focus on μ_3 (see Zhang's thesis for the general case). Our main theorem is the following:

Theorem 1 (Distance Distortion Range for μ_3).

$$D_{\mu_3} = [\tfrac{1}{4}, 1].$$

Thus the distortion constant for μ_3 is 4.

The key to the proof lies in exploiting the Riemannian metric of S^3 and $\widehat{S^3}$. In practical applications, we want representations whose distortion constant is as small as possible, subject to other considerations. E.g., our SSS planner [10] (like many algorithms) uses an $\varepsilon > 0$ parameter to discard a subdivision box B if "$\varepsilon > width(B)$". Clearly $width(B)$ is a distorted substitute for distance in $SO(3)$, but how distorted is it?

The following simple lemma is very useful:

Lemma 1 (Composition of Distortion Range). *If $f : X \to Y$ and $g : Y \to Z$ have distortion ranges of $[a, b]$ and $[c, d]$ (respectively), then the distortion range of $h = g \circ f : X \to Z$ is contained in $[ca, db]$. If $c = d$, then $D_h = [ca, cb]$.*

See proof in [11]. To apply this lemma, let f be the map μ_3 and for $K > 0$, let $g_K : \partial[-K, K]^3 \to \partial[-1, 1]^3$ where $g_K(q) = q/K$. So g_K has distortion range $[\frac{1}{K}, \frac{1}{K}]$. Then Lemma 1 implies that the map $g_K(\widehat{\cdot}) : \partial[-K, K]^3 \to S^3$ has distortion range $[1/4K, 1/K]$. Hence the distortion constant becomes $C_0 = \max\{1/K, 4K\}$ (see (3)). By choosing K to minimize the distortion constant, this proves:

Theorem 2 (Parametric Cubic Models). *Consider representations*

$$\mu_{n,K} : \partial[-K, K]^n \to S^n$$

which are parametrized by $K > 0$.

(a) ($n = 3$) The optimal distortion of $C_0 = 2$ is achieved when $K = 1/2$.
(b) ($n = 2$) The optimal distortion of $C_0 = \sqrt{3}$ is achieved when $K = 1/\sqrt{3}$.

In practice, we would like K to be a dyadic number (BigFloats) so that subdivision (which is typically reduced to bisection) can be carried out exactly. This implies we should choose $K = 1/2$ when $n = 2$ to get a suboptimal distortion of $C_0 = 2$. This remark is relevant for our 5DOF robots in [2].

3 Reduction to Distortion in Riemannian Metric

We now reduce the distortion of maps between metric spaces to the distortion of diffeomorphisms F between Riemannian manifolds M, N,

$$F : M \to N$$

where M, N are smooth n-dimensional manifolds. Our terminology and notations in differential geometry follow [7,5]. A Riemannian manifold is a pair (M, g_M) where g_M (called a **Riemannian metric**) is an inner product (positive, bilinear, symmetric) on vectors $u_{\boldsymbol{p}}, v_{\boldsymbol{p}}$ in the tangent space $T_{\boldsymbol{p}}M$ ($\boldsymbol{p} \in M$). We write "$g_M\langle u_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle$" instead of $g_M(u_{\boldsymbol{p}}, v_{\boldsymbol{p}})$ to suggest the inner product property. Also, write $|u_{\boldsymbol{p}}|_{g_M} := \sqrt{g_M\langle u_{\boldsymbol{p}}, u_{\boldsymbol{p}}\rangle}$. As (N, g_N) is also a Riemannian manifold, we get an (induced) Riemannian metric $F^*(g_N)$ for M where

$$F^*(g_N)\langle u_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle := g_N\langle D_F(u_{\boldsymbol{p}}), D_F(v_{\boldsymbol{p}})\rangle, \tag{4}$$

called the F**-pullback** of g_N [5, p.333], and D_F is the Jacobian of F. Then we define the **metric distortion range** of F, namely, the closure of the set

$$MD_F = [m_F, M_F] := \left\{ \frac{|v_{\boldsymbol{p}}|_{F^*(g_N)}}{|v_{\boldsymbol{p}}|_{g_M}} : v_{\boldsymbol{p}} \in T_{\boldsymbol{p}}M, \boldsymbol{p} \in M \right\}. \tag{5}$$

From any Riemannian manifold (M, g_M), we derive a distance function[5] given by

$$d_{g_M}(\boldsymbol{p}, \boldsymbol{q}) := \inf_{\boldsymbol{p} \overset{\pi}{\rightsquigarrow} \boldsymbol{q}} \int_0^1 |\pi'(t)|_{g_M} dt = \inf_{\boldsymbol{p} \overset{\pi}{\rightsquigarrow} \boldsymbol{q}} \int_0^1 \sqrt{g_M\langle \pi'(t), \pi'(t)\rangle} dt \tag{6}$$

where $\boldsymbol{p} \overset{\pi}{\rightsquigarrow} \boldsymbol{q}$ means that $\pi : [0,1] \to M$ is a smooth curve with $\pi(0) = \boldsymbol{p}$, $\pi(1) = \boldsymbol{q}$; also $\pi'(t)$ is the tangent vector to the curve at $\pi(t)$. The pair (M, d_{g_M}) is now a metric space [5, Theorem 13.29, p. 339].

The **distance distortion range** of $F : (M, d_{g_M}) \to (N, d_{g_N})$ (viewed as maps between metric spaces) is

$$D_F := \left\{ \frac{d_{g_N}(F(\boldsymbol{p}), F(\boldsymbol{q}))}{d_{g_M}(\boldsymbol{p}, \boldsymbol{q})} : \boldsymbol{p}, \boldsymbol{q} \in M, \boldsymbol{p} \neq \boldsymbol{q} \right\}. \tag{7}$$

We next connect distance distortion to metric distortion of F:

Theorem 3 (Metric Distortion).
If $F : M \to N$ is a smooth map between two Riemannian manifolds, then the metric distortion range of F is equal to the distance distortion range of F:

$$MD_F = D_F$$

See proof in [11].

3.1 Metric Distortion Range of μ_3

Our Theorem 1 is now a consequence of Theorem 3 and the following theorem.

[5] Following [5, p. 328], we call d_{g_M} a **distance function**, reserving "metric" for g_M.

Theorem 4 (Metric Distortion Range for μ_3).
The metric distortion range MD_{μ_3} for $\mu_3 : \widehat{SO}_3 \to S^3$ is

$$[m_{\mu_3}, M_{\mu_3}] = [\tfrac{1}{4}, 1].$$

Moreover, there exists $\boldsymbol{p}, \boldsymbol{q} \in \widehat{SO}_3$, and $\boldsymbol{u} \in T_{\boldsymbol{p}}\widehat{SO}_3$, $\boldsymbol{v} \in T_{\boldsymbol{q}}\widehat{SO}_3$ such that

$$m_{\mu_3} = \frac{|\boldsymbol{u}|_{g_{S^3}}}{|\boldsymbol{u}|_{g_{\widehat{SO}_3}}}, \qquad M_{\mu_3} = \frac{|\boldsymbol{v}|_{g_{S^3}}}{|\boldsymbol{v}|_{g_{\widehat{SO}_3}}}. \tag{8}$$

Note that our theorem gives the exact metric distortion range. To achieve this, we will find two expressions for the metric norm $|v_{\boldsymbol{p}}|_{g_N}$: one that achieves the upper bound, and another that achieves the lower bound.

Proof. Let $\mu_3 : \partial[-1,1]^4 \to S^3$ where $\partial[-1,1]^4$ is viewed as the union of 8 cubes, corresponding to each choice of $w, x, y, z = \pm 1$. By symmetry, we focus on the cube $B_1 = \{(w,x,y,z) \in \partial[-1,1]^4 : w = 1\}$. Thus

$$\mu_3(1,x,y,z) = \tfrac{1}{r}(1,x,y,z)$$

where $r = \sqrt{1+x^2+y^2+z^2}$. If g_{S^3} is the induced Riemannian metric for S^3, then MD_{μ_3} is the range of $\sqrt{\frac{\mu_3^*(g_{S^3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle}{g_{B_1}\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle}}$ over $v_{\boldsymbol{p}} \in T_{v_{\boldsymbol{p}}}B_1$ for all $\boldsymbol{p} \in B_1$ and $\mu_3^*(g_{S^3})$ is pull-back metric. First compute the Jacobian of μ_3:

$$D_{\mu_3} = J_{\mu_3} = \begin{pmatrix} \frac{\partial(1/r)}{\partial x} & \frac{\partial(1/r)}{\partial y} & \frac{\partial(1/r)}{\partial z} \\ \frac{\partial(x/r)}{\partial x} & \frac{\partial(x/r)}{\partial y} & \frac{\partial(x/r)}{\partial z} \\ \frac{\partial(y/r)}{\partial x} & \frac{\partial(y/r)}{\partial y} & \frac{\partial(y/r)}{\partial z} \\ \frac{\partial(z/r)}{\partial x} & \frac{\partial(z/r)}{\partial y} & \frac{\partial(z/r)}{\partial z} \end{pmatrix} = \frac{1}{r^3}\begin{pmatrix} -x & -y & -z \\ x^2-r^2 & xy & xz \\ xy & y^2-r^2 & yz \\ xz & yz & z^2-r^2 \end{pmatrix}. \tag{9}$$

In the following, we may assume that $v_{\boldsymbol{p}} = (\mathrm{dx}, \mathrm{dy}, \mathrm{dz})^\mathsf{T} \in T_{\boldsymbol{p}}B_1$ where

$$g_{B_1}(v_{\boldsymbol{p}}, v_{\boldsymbol{p}}) = \langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle = \mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2 = 1. \tag{10}$$

The pull-back metric
$$\begin{aligned} \mu_3^*(g_{S^3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle &= \big\langle J_{\mu_3}\cdot v_{\boldsymbol{p}}, J_{\mu_3}\cdot v_{\boldsymbol{p}}\big\rangle \\ &= v_{\boldsymbol{p}}^\mathsf{T}(J_{\mu_3}^\mathsf{T}\cdot J_{\mu_3})v_{\boldsymbol{p}} \\ &= \frac{1}{r^4}v_{\boldsymbol{p}}^\mathsf{T}\cdot\begin{pmatrix} r^2-x^2 & -xy & -xz \\ -xy & r^2-y^2 & -yz \\ -xz & -yz & r^2-z^2 \end{pmatrix}\cdot v_{\boldsymbol{p}} \quad \text{(from (9))} \\ &= \frac{E}{r^4} \end{aligned} \tag{11}$$

where

$$E := (r^2-x^2)\mathrm{dx}^2 + (r^2-y^2)\mathrm{dy}^2 + (r^2-z^2)\mathrm{dz}^2 - 2\big(xy\mathrm{dxdy} + yz\mathrm{dydz} + xz\mathrm{dxdz}\big)$$

To facilitate further manipulation, rewrite E in the compact form

$$E = \sum_i (r^2 - x_i^2)\mathrm{dx}_i^2 - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \tag{12}$$

where $(x, y, z) = (x_1, x_2, x_3)$ and the sums $\sum_i, \sum_{i,j}$ (and $\sum_{i,j,k}$) are interpreted appropriately: i, j, k range over $\{1, 2, 3\}$, with i chosen independently, but j chosen to be different from i, and k chosen to be different from i and j. Thus each sum has exactly 3 summands.

To obtain upper and lower bounds on $\mu_3^*(g_{S^3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle = E/r^4$, we need to express E in two different ways:

(A) For the lower bound,

$$\begin{aligned}
E &= \sum_i (r^2 - x_i^2)\mathrm{dx}_i^2 - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \\
&= \sum_{i,j,k} (1 + x_j^2 + x_k^2)\mathrm{dx}_i^2 - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \\
&= \sum_i \mathrm{dx}_i^2 + \sum_{i,j} (x_i^2 \mathrm{dx}_j^2 + x_j^2 \mathrm{dy}_i^2) - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \\
&= 1 + \sum_{i,j} (x_i \mathrm{dx}_j - x_j \mathrm{dx}_i)^2 \qquad \left(\text{as } 1 = \sum_i \mathrm{dx}_i^2\right)
\end{aligned}$$

Hence,

$$\mu_3^*(g_{S^3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle = \frac{1 + \sum_{i,j}(x_i \mathrm{dy}_j - y_j \mathrm{dx}_i)^2}{r^4} \geqslant \frac{1}{(1 + x^2 + y^2 + z^2)^2} \geqslant \frac{1}{16}.$$

The last two inequalities become equalities when $x = y = z = 1$ and $\mathrm{dx} = \mathrm{dy} = \mathrm{dz} = 1/\sqrt{3}$. This proves a tight lower bound of $1/16$ for $\mu_3^*(g_{S_3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle$.

(B) To obtain an upper bound, we rewrite E as follows:

$$\begin{aligned}
E &= \sum_{i,j,k} (1 + x_j + x_k)\mathrm{dx}_i^2 - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \\
&= \sum_i \mathrm{dx}_i^2 + \sum_{i,j,k} x_i^2 (\mathrm{dx}_j^2 + \mathrm{dx}_k^2) - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \\
&= 1 + \sum_i x_i^2 (1 - \mathrm{dx}_i^2) - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \qquad \left(\text{as } 1 = \sum_i \mathrm{dx}_i^2\right) \\
&= \left(1 + \sum_i x_i^2\right) - \sum_i x_i^2 \mathrm{dx}_i^2 - 2\sum_{i,j} x_i x_j \mathrm{dx}_i \mathrm{dx}_j \\
&= r^2 - \left(\sum_i x_i \mathrm{dx}_i\right)^2
\end{aligned}$$

$$\mu_3^*(g_{S_3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle = \frac{r^2 - (x\mathrm{dx} + y\mathrm{dy} + z\mathrm{dz})^2}{r^4} \leqslant \frac{1}{r^2} \leqslant 1.$$

The last two inequalities are equalities when $x = y = z = 0$.

We have therefore established that

$$\sqrt{\mu_3^*(g_{S_3})\langle v_{\boldsymbol{p}}, v_{\boldsymbol{p}}\rangle} \in [\tfrac{1}{4}, 1]$$

and these bounds are achievable. **Q.E.D.**

With the above compact notation, we could generalize the argument to μ_n (see [11]), showing $MD_{\mu_n} = [1/(n+1), 1]$.

4 Final Remarks

1. It remains to determine the distortion of the representation $\overline{\mu}_3 : \widehat{S^3} \to SO(3)$, where $\overline{\mu}_3 = \mu_3 \circ \rho$ and $\rho : S^3 \to SO(3)$ is the usual map from unit quaternions to $SO(3)$. Huynh observed that $\Phi_6(\rho(q_1), \rho(q_2)) = 2\Phi_3(q_1, q_2)$ [3]. Since Φ_3 and Φ_6 are natural distance functions on S^3 and $SO(3)$, the distortion range of ρ is $[2, 2]$. By Lemma 1, we conclude that the distance distortion of $\overline{\mu}_3$ is

$$D_{\overline{\mu}_3} = 2[\tfrac{1}{4}, 1] = [\tfrac{1}{2}, 2].$$

2. This work establishes the exact distortion bounds on the cubic model representation $\overline{\mu}_3$ of $SO(3)$. A critical step was to exploit Riemannian geometry by reducing distance distortion to metric distortion. We expect our cubic representation to have other applications, e.g., a rigorous subdivision search for an ε-optimal rotation to best fit experimental data as in motion capture research.

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