Homework 5

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Be sure to read my Notes on Eigenvalues and Singular Values before attempting this homework. You may find it helpful to experiment with MATLAB while doing the homework, but be sure to explain *why* statements are true, not just say "MATLAB says so". As before, if you are stuck on some questions, consult with other students or the web, remembering to acknowledge any help you get, or come to my office hours or send email to make an appointment with me.

1. Prove the following statements, using the basic definition of eigenvalues and eigenvectors, **or** give a counterexample showing the statement is not true. Assume A is an $n \times n$ real matrix.

   (a) If λ is an eigenvalue of A and $\alpha \in \mathbb{R}$, then $\lambda + \alpha$ is an eigenvalue of $A + \alpha I$, where I is the identity matrix.

   (b) If λ is an eigenvalue of A and $\alpha \in \mathbb{R}$, then $\alpha\lambda$ is an eigenvalue of αA.

   (c) If λ is an eigenvalue of A, then for any positive integer k, λ^k is an eigenvalue of A^k. (Here A^k means the matrix product $AA \cdots A$, where A appears in the product k times.)

   (d) If B is "similar" to A, which means that there is a nonsingular matrix S such that $B = SAS^{-1}$, then if λ is an eigenvalue of A it is also an eigenvalue of B.

   (e) Every matrix has at least two distinct eigenvalues, say λ and μ, with $\lambda \neq \mu$.

   (f) Every real matrix has a real eigenvalue.

   (g) If A is singular, then it has an eigenvalue equal to zero.

   (h) If A is singular, then it has a singular value equal to zero.

   (i) If all the eigenvalues of a matrix A are zero, then $A = 0$.

   (j) If all the singular values of a matrix A are zero, then $A = 0$.

2. The determinant has the property that for any two square matrices X and Y, $\det(XY) = \det(X)\det(Y)$, and that the determinant of a diagonal matrix is the product of its diagonal entries. Assuming A is diagonalizable, show using its eigenvalue decomposition that the determinant of a square matrix A is the product of its eigenvalues. (This is true for all square matrices but it's more complicated to show in general.)

3. Another property of the determinant is that if X is a square real matrix, $\det(X^T) = \det(X)$. Use this to show that if Q is orthogonal, which means $Q^T Q = I$, then $\det(Q) = \pm 1$. Then use the SVD to show that the determinant of a square real matrix A is **plus or minus** the product of its singular values.

4. Let A be a square real matrix, say $n \times n$, with $A \neq A^T$, with SVD given by $A = U\Sigma V^T$, with rank r.

   (a) What are the singular values and the left and right singular vectors of the transposed matrix A^T, in terms of Σ, U and V?

   (b) What is the rank of A^T?

   (c) Give an orthonormal basis for the range of A^T, in terms of U and/or V.

   (d) Give an orthonormal basis for the null space of A^T, in terms of U and/or V.

   (e) What does this tell you about the relationship between the range of A and the null space of A^T?

5. Let the real square $n \times n$ matrix A have SVD given by $A = U\Sigma V^T$.

   (a) What are the eigenvalues and eigenvectors of the real symmetric matrix $B = A^T A$ in terms of Σ, U and V?

   (b) What are the eigenvalues and eigenvectors of the real symmetric matrix $B = AA^T$ in terms of Σ, U and V?

6. Using MATLAB, investigate the relationship between the eigenvalues and eigenvectors of the real symmetric matrix

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

   and the singular values and singular vectors of A. Once you see the pattern, explain why it is true.

7. Confirm the plot at the top of p. 4 of the notes by writing a MATLAB code to generate vectors defining points on the unit circle in two dimensions (using `cos` and `sin`), and, given a randomly generated 2×2 matrix A, multiply A onto these vectors and plot the resulting vectors (they should trace out an ellipse). Use as many vectors as you need to make a nice picture. Also, draw v_1, v_2 on the circle plot and $\sigma_1 u_1, \sigma_2 u_2$ on the ellipse plot as shown on p. 4 (compute these first from `svd`). Submit printed copies of the MATLAB code and the plots.

8. Assume A is an $n \times n$ real matrix and consider the Schur decomposition $A = QUQ^T$, or equivalently $AQ = QU$, where Q is orthogonal and U is quasi-upper triangular. If A is symmetric, i.e. $A = A^T$, what special form does U have? Are the columns of Q eigenvectors (why or why not?)

9. Considering again the Schur decomposition, suppose that A is not symmetric, but that its eigenvalues are all real, so that U is upper triangular, with no 2×2 blocks on the diagonal. Then, it turns out that one of the Schur vectors (one of the columns of Q) is an eigenvector. Which one and why? Remember that eigenvectors are only defined up to a scalar multiplication, so the Schur vector which is an eigenvector delivered by `schur` might not look like an eigenvector delivered by `eig`, unless you normalize them consistently.