1. Consider a linear system of algebraic equation $Ax = b$. Here the matrix $A$ has three rows and four columns.

(a) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.

(b) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.

(c) Formulate, if possible, necessary and sufficient conditions on $A$ and $b$ which guarantee that at least one solution exists.

(d) Formulate, if possible, necessary and sufficient conditions on $A$ which guarantee that at least one solution exists for any choice of $b$.

2. A real matrix $A$ is said to skew-symmetric if $A^T = -A$. Prove that such a matrix has a full set of eigenvectors and that all its eigenvalues are purely imaginary, i.e., are of the form $ai$ where $a$ is a real number and $i^2 = -1$.

3. Consider $<f, g> = \int_{-1}^{+1} f(t)g(t)dt$.

Here $f(t)$ and $g(t)$ are continuous functions.

(a) Show that this defines an inner product space $V$ over the complex field.

(b) Let the span of $1, t$, and $t^2$ define a subspace of $V$. Create an orthonormal basis of this subspace.

(c) Find the second order polynomial which is closest to the function $t^3$ in the norm defined by the inner product defined above.
4. In Gaussian elimination, three matrices $P$, $L$, and $U$ are computed for a given square matrix $A$. Here $P$ represents a permutation of the rows of $A$, $L$ is lower triangular with diagonal elements all $= 1$, and $U$ is upper triangular and $PA = LU$.

(a) What is partial pivoting?
(b) How can $P$, $L$, and $U$ be used to solve a linear system of equations $Ax = b$?
(c) Can this algorithm ever fail if we use exact arithmetic?
(d) What characterizes a matrix $A$ for which the solution of $Ax = b$ is very sensitive to small changes in $b$?
(e) How does the work grow as a function of $n$ where $A$ is $n$ by $n$.

5. Show that the rank of any matrix is unchanged if it is multiplied from the left or the right by a square, nonsingular matrix of the appropriate size.

6. Consider a square matrix of order $n$ defined by $I - 2vv^T$. Here $v$ is column vector with $n$ real components.

(a) Under what condition is this a matrix with orthonormal columns?
(b) Give the geometric context of this type of transformation.
(c) How are such matrices used to solve linear least squares problems?