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GAMES

MATHEMATICIANS PLAY

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THE TENURE GAME

Each year, Chair Paul gives promotion list \( L \) to Dean Carole. Carole Either

- Promotes \( L \), Fires \( \bar{L} \) or
- Promotes \( \bar{L} \), Fires \( L \)

Carole wins if nobody gets tenure.
$a_k$ people $k$ rungs from Tenure

Theorem. If $\sum a_k 2^{-k} < 1$ then Carole wins.


$T =$ number getting Tenure.

$\Pr[\text{Paul wins}] \leq E[T] = \sum a_k 2^{-k} < 1$

Therefore Carole can always win.

Proof2. (Derandomization)

Carole plays to minimize $E[T]$.
Theorem. If $\sum a_k 2^{-k} \geq 1$ then Paul wins.

Lemma. If $E[T] \geq 1$ there is a move for Paul so that $E[T^{yes}] \geq 1$ and $E[T^{no}] \geq 1$.

Proof of Theorem:

Paul makes that splitting move.
BALANCING VECTOR GAME

$n$ rounds. Initial $P \leftarrow 0 \in \mathbb{R}^n$

Paul picks $v_i \in \{-1, +1\}^n$

Carole picks $\epsilon_i \in \{-1, +1\}$

$P \leftarrow P + \epsilon_i v_i$

Payoff to Paul: $|P_{final}|_\infty$

$VAL(n)$: value of Game.

Similar to:

- On Line Coloring of $A_1, \ldots, A_n \subseteq \{1, \ldots, n\}$
- On Line Roundoff of $x_1, \ldots, x_n \in [0, 1]$ to minimize max error in linear $L_1, \ldots, L_n$

Carole $\sim$ Worst Case Analysis
Theorem. If

\[ \Pr[|S_n| > \alpha] < n^{-1} \]

then Carole can keep \( |P_{final}|_\infty < \alpha \)

Proof1. Carole plays randomly

\[ T = \text{number of coordinates } L_i \text{ with } |L_i| > \alpha \]

\[ E[T] = n \Pr[|S_n| > \alpha] < 1 \]

\[ \Pr[\text{Paul wins}] \leq E[T] < 1 \]

Therefore Carole can always win

Proof2 (Derandomization)

\[ P = (L_1, \ldots, L_n) \text{ with } t \text{ rounds remaining.} \]

\[ E[T] = w_t(P) = \sum \Pr[|L_i + S_t| > \alpha] \]

Carole plays to minimize \( E[T] \)

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Theorem. If
\[ \Pr[|S_n| > \alpha] > cn^{-1/2} \]
then Paul can force \(|P_{final}|_\infty > \alpha\)

Proof. With \(t+1\) rounds remaining Paul picks \(v = (\delta_1, \ldots, \delta_n)\) with
\[ |w_t(P + v) - w_t(P - v)| \leq \]
\[ \leq \max |\Pr[|L_i+1+S_t| > \alpha] - \Pr[|L_i-1+S_t| > \alpha]| \]
\[ = O(t^{-1/2}) \]

Then \(w(P^{new}) > w(P^{old}) - O(t^{-1/2})\)
\(w(P_{final}) > w(P_{init}) - \sum O(t^{-1/2}) >\)
\[ > w(P_{init}) - O(n^{1/2}) > 0 \]

Corollary. \(VAL(n) = \Theta(\sqrt{n \ln n})\)
PAUL AND CAROLE GAMES

• RANDOMIZATION
   Carole plays randomly. If she wins with positive probability she can always win.

• DERANDOMIZATION
   Conditional Expectation gives weight function for Carole to minimize deterministically.

• ANTIRANDOMIZATION
   Paul uses *this* weight function for effective counterplay.
LIAR GAME \((n, q, k)\)

Paul must guess \(x \in \{1, \ldots, n\}\) with \(q\) Yes/No questions.

Carole can lie at most \(k\) times.

Carole doesn’t really select \(x\).

CHIP VERSION:

\(x_i\) chips on square \(i\), \(0 \leq i \leq k\)

\(q\) rounds

Paul picks subset of chips.

Carole moves those or complement up one square.

Carole wins if at least two chips survive.
Theorem. If with $t$ rounds remaining
\[ \sum x_i \Pr[B(t,.5) \leq k - i] > 1 \]
then Carole wins.

Proof1. Carole plays randomly
$T =$ number of chips surviving
\[ \Pr[\text{Paul wins}] = \Pr[T \leq 1] \leq E[T] < 1 \]
so Carole can always win

Proof2 (Derandomization)
Carole plays to maximize $E[T]$

Corollary. Carole wins if
\[ n > \frac{2^q}{1 + \left(\frac{q}{1}\right) + \left(\frac{q}{2}\right) + \ldots + \left(\frac{q}{k}\right)} \]
$k$ fixed, $q > q_0(k)$, $c = c(k)$

Theorem (JS). If

$$
\sum x_i \Pr[B(q, .5) \leq k - i] \leq 1
$$

and $x_k > cq^k$ then Paul wins

Idea: Assume wlog $\sum = 1$. Show that Paul can always find perfect split.

Corollary. Necessary and Sufficient asymptotic conditions for Paul to win LIAR GAME $(n, q, k)$