1. Suppose $n \geq 2$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size $n$. Suppose $m < 4^{n-1}$. Show that there is a coloring of $\Omega$ by 4 colors so that no edge is monochromatic.

Solution. In a random coloring an $A_i$ is monochromatic with probability $\frac{4}{1-4^{-n}}$ so with $m < 4^{n-1} < 1$ there is a coloring with none of the bad events.

2. Suppose $n \geq 4$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size $n$. Suppose $m < \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of $\Omega$ by 4 colors so that in every edge all 4 colors are represented.

Solution. In a random coloring an $A_i$ has three (or fewer) colors with probability at most $\frac{4}{3} \cdot \frac{3}{4} n$ (the 4 being the choice of missing color and $\frac{3}{4} n$ the probability it is missing). Here $m \cdot \frac{4}{3} \cdot \frac{3}{4} n < 1$ so the argument is as before.

3. The expected number of isolated trees [just take this as a fact] on $k$ vertices in $G(n, p)$ is given by $f(n, k, p) := \binom{n}{k} k^{k-2} p^{k-1} (1-p)^B$ with $B = k(n-k) + \binom{k}{2} - k + 1$. Set $p = \frac{1}{n}$. Let $c$ be a positive constant. Find the asymptotics of $f(n, k, p)$ when $k \sim cn^{2/3}$. (*) Express the limit as $n \to \infty$ of the sum of $f(n, k, p)$ for $n^{2/3} \leq k < 2n^{2/3}$ as a definite integral and use a computer package to evaluate the integral numerically.

The solution is given on P168-169 (section 10.6) of the text.

4. Consider Boolean expressions on atoms $x_1, \ldots, x_n$. By a $k$-clause $C$ we mean an expression of the form $y_{i1} \lor \ldots \lor y_{ik}$ where each $y_{ij}$ is either $x_{ij}$ or $\overline{x}_{ij}$. Prove a theorem of the following form [you fill in the $m = m(k)$] by the probabilistic method: For any $m$ $k$-clauses $C_1, \ldots, C_m$ there is a truth assignment such that $C_1 \land \ldots \land C_m$ is satisfied.

Solution. When $m < 2^k$ this holds. Take a random truth assignment. For $1 \leq i \leq m$ let $BAD_i$ be that $C_i$ is not satisfied so $\Pr[BAD_i] = 2^{-k}$ and $m2^{-k} < 1$ so with positive probability the truth assignment satisfies all $C_i$ and hence $C_1 \land \ldots \land C_m$.

Working with Paul Erdős was like taking a walk in the hills. Every time I thought that we had achieved our goal and
deserved a rest, Paul pointed to the top of another hill and off we would go.
– Fan Chung