Random Graphs G22.3033-007

Assignment 1. Solutions

1. The Bipartite Ramsey Number \( BR(k) \) is the least \( n \) so that if \( A, B \) are disjoint with \(|A| = |B| = n\) and \( A \times B \) is two colored there exist \( A_1 \subseteq A, B_1 \subseteq B \) with \(|A_1| = |B_1| = k\) and \( A_1 \times B_1 \) monochromatic. Find and prove a theorem which gives a lower bound for \( BR(k) \) and explore the asymptotics.

Solution. There are \( \binom{n}{k}^2 \) possible choices for \( A_1, B_1 \) and each has probability \( 2^{-k^2} \) of being monochromatic so the theorem would be:

\[
\text{If } \binom{n}{k}^2 2^{1-k^2} < 1 \text{ then } BR(k) > n
\]

For the rough asymptotics, using \( \binom{n}{k} \leq n^k \), it suffices that \( n^{2k} 2^{1-k^2} < 1 \) or \( n < 2^{k/2}(1 - o(1)) \) so \( BR(k) > 2^{k/2}(1 + o(1)) \). More precisely, as \( n \) is exponential in \( k \) we have \( \binom{n}{k} \sim n^k/k! \) so it suffices that \( n^{2k} 2^{1-k^2} < k!^2 \). Taking \( 2k \)-th roots we have \( (k!)^2/2^{k^2} = k!^{1/k} \sim k/e \) so \( BR(k) < (k/e)^{2^{k^2}/(1 + o(1))} \).

2. Let \( f(k) \) be the maximal \( n \) for which there exists \( p \) with \( 0 \leq p \leq 1 \) such that

\[
n^k p^{k^2/2} + n^{2k} (1 - p)^{2k^2} \leq 1
\]

Let \( U(k) \) be the maximal \( n \) for which there exists \( n^k p^{k^2/2} \leq 1 \) and \( n^{2k} (1 - p)^{2k^2} \leq 1 \). Let \( L(k) \) be the maximal \( n \) for which there exists \( p \) with \( n^k p^{k^2/2} \leq 1/2 \) and \( n^{2k} (1 - p)^{2k^2} \leq 1/2 \).

(a) Argue that \( L(k) \leq f(k) \leq U(k) \)

Solution. When both terms are at most one half their sum is tautologically at most one. For the sum to be at most one it is necessary that both terms be at most one.

(b) Find the asymptotics of \( U(k) \). (Warning: Do not assume \( p = o(1) \) because the optimal \( p \) isn’t!) Partial credit for \( \lim_k U(k)^{1/k} \).

Solution: Taking appropriate roots we need \( n \leq p^{-k/2} \) and \( n \leq (1-p)^{-k} \) so \( U(k) = \max(p, (1-p)^2)^{-k/2} \). The functions \( p, (1-p)^2 \) are increasing and decreasing respectively in \( p \) (in \([0,1]\)) so their max is achieved when they are equal, \( p = \frac{1}{2}(3 - \sqrt{5}) \) and \( U(k) = (\frac{1}{2}(3 - \sqrt{5}))^{-k/2} \). But its more clear to write it \( U(k) \leq \phi^k \) where \( \phi := \frac{1}{2}(1 + \sqrt{5}) = 1.61 \ldots \) is the golden ratio!
(c) Find the asymptotics of \(L(k)\), showing that it is the same as that of \(U(k)\). (That is, changing 1 to \(\frac{1}{2}\) had an asymptotically negligible effect.) Solution: Suppose \(n_0, p\) work for the upper bound, that is, both terms are at most one. Let \(n = n_0 2^{-1/k}\). Now \(n, p\) works for the lower bound, as \(n^k\) and \(n^{2k}\) have gone down by factors of two and four respectively. While we can’t say this is the optimal \(n\), since it works it gives a lower bound and so \(L(k) \geq n_0 2^{-1/k}\). Take \(n_0 = U(k)\), optimal for the upper bound. So \(L(k) \geq U(k) 2^{-1/k}\). As \(\lim k^{2^{-1/k}} = 1\), \(U(k) \sim L(k)\).

(d) Deduce the asymptotics of \(f(k)\)
Solution: \(f(k) \sim \phi k^{3}\).

3. Find asymptotic lower bounds on the Ramsey function \(R(k, 2k)\). That is, set \(g(k)\) to be the maximal \(n\) for which there exists \(p\) with \(0 \leq p \leq 1\) such that

\[
\left(\frac{n}{k}\right)^{p(k)} + \left(\frac{n}{2k}\right)^{(1 - p)} < 1
\]

Find an asymptotic formula for \(g(k)\). (Note: You’ll want to use the ideas of the previous problem. Still, this is not an easy problem. Full marks for \(\lim k g(k)^{1/k}\) – it’s the same as \(\lim k f(k)^{1/k}\) but you have to prove this. The full asymptotics are if you enjoy a challenge.)
Partial Solution: Let \(U^*(k), L^*(k)\) be the bounds when one requires both terms at most one and both terms at most one half respectively. The argument previously used shows these are asymptotic so it suffices to look at \(U^*(k)\). Taking appropriate roots

\[
U^*(k) = (1 + o(1)) \max_p \left(\frac{k}{e} p^{-(k-1)/2}, \frac{2k}{e} (1 - p)^{-(2k-1)/2}\right)
\]

so the optimal \(p\) is when

\[
\frac{k}{e} p^{-(k-1)/2} = \frac{2k}{e} (1 - p)^{-(2k-1)/2}
\]

Now \(p \sim p_0 = \frac{1}{2}(3 - \sqrt{5})\). For more precise results one (this is the art) parametrizes \(p = p_0 + \frac{x}{k}\) and finds the constant \(x\) that gives asymptotic equality. One then plugs that \(p\) back to get \(U^*(k)\).

4. Find \(m = m(n)\) as large as you can so that the following holds: Let \(A_1, \ldots, A_m \subseteq \{1, \ldots, 4n\}\) with all \(|A_i| = n\). Then there exists a two
coloring of \{1, \ldots, 4n\} such that none of the \(A_i\) are monochromatic. Use a random equicoloring of \{1, \ldots, 4n\}. Express your answer as an asymptotic function of \(n\).

Solution. For a random equicoloring any \(n\)-set \(A_i\) is monochromatic with probability \(p := \frac{2^{3n}}{4n^2}\). [The 2 factor is Red/Blue choice, the \(\binom{3n}{n}\) is the number of colorings with \(A_i\) Red, the \(\binom{4n}{2n}\) is the total number of colorings. So if \(mp < 1\) there exists a coloring. Thus we can take \(m = \lfloor p^{-1} \rfloor\). Stirling’s Formula gives then \(m(n) \geq (64/27 + o(1))^n\), ignoring lower order terms. Note this is significantly better than the bound \(m(n) > 2^{n-1}\) that one has without the assumption on the universe size.