Algebra, Assignment 9
Due, Monday, Nov 13

Note: When we write $\mathbb{Z}_n$ we assume the operation is $+$.

1. (a) Find a subgroup of $\mathbb{Z}_{5^9} \times \mathbb{Z}_{5^8}$ of size $5^{14}$.
   (b) Generalize to show that if $G$ is Abelian and $o(G) = p^w$ and $1 \leq s < w$ then there is a subgroup $H$ of $G$ of size $p^s$. (Use the characterization theorem of Finite Abelian Groups.)
   (c) Generalize to show that if $G$ is Abelian and $o(G) = n$ and $m|n$ then there is a subgroup $H$ of size $m$

2. Let $\mathbb{Z}[i]$ denote \{a + bi : a, b \in \mathbb{Z}\} where $i = \sqrt{-1}$. This structure is called the Gaussian Integers and will be a frequent example.
   (a) Show that $\mathbb{Z}[i]$ contains 0, 1, and is closed under addition, additive inverse, and multiplication.
   (b) Precisely which elements of $\mathbb{Z}[i]$ have multiplicative inverses?
      (Note: It is not enough that they have multiplicative inverses in the complex numbers, their multiplicative inverse must be in $\mathbb{Z}[i]$ itself.)
   (c) Define $\phi : \mathbb{Z}[i] \to \mathbb{Z}[i]$ by $\phi(a + bi) = a - bi$. (This is generally known as complex conjugation.) Show that $\phi$ is a homomorphism by showing $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$. Show that $\phi$ has kernel \{0\}.
   (d) A number $\alpha \in \mathbb{Z}[i]$ is called composite if we can write $\alpha = \beta\gamma$ where neither $\beta$ nor $\gamma$ have multiplicative inverses. (That last condition is to avoid “trivial” factorizations like $23 = (-1)(-23).$)
      Show that 2 is composite. Show that 41 is composite.

Bach, Mozart, Schubert - they will never fail you. When you perform their work properly it will have the character of the inevitable, as in great mathematics, which seems always to be made of pre-existing truths.

E. L. Doctorow, *City of God*