ALGEBRA FINAL

Originality and a feeling of one’s own dignity are achieved only through work and struggle. – Dostoyevsky

No books or notes. Problems marked (−) are relatively easy while those marked (*) are particularly challenging. Maximal Grade 220.

DO THREE OF PROBLEMS 1,2,3,4

1. (15) Assume as a fact that when \( p \) is a prime \( \mathbb{Z}_p^* \) is cyclic. Assume as a fact that 701 is prime. How many \( x \in \mathbb{Z}_{701}^* \) satisfy \( x^7 = 1 \)? (You must give a clear reason for your answer!)

2. (15) The center \( Z \) of a group \( G \) is the set of all \( z \in G \) with the property that \( zg = gz \) for all \( g \in G \). Prove that \( Z \) is a subgroup of \( G \).

3. (15) In \( S_3 \) let \( H = \{e, \sigma\} \) with \( \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \) List the right cosets of \( H \).

4. (15) Let \( G, g \in G \) and define \( \phi : G \to G \) by \( \phi(x) = gxg^{-1} \). Show that \( \phi \) is a homomorphism.

5. (20) Let \( G \) be an Abelian Group under multiplication and define \( \phi : G \to G \) by \( \phi(x) = x^3 \). Show that \( \phi \) is a homomorphism. Now assume further that \( G \) is a finite group and that \( o(G) \) is not divisible by 3. Show that \( \phi \) is a bijection.

6. (25) With \( D_{10} \) as given in the attached sheet:
   (a) (−)(5) What is the inverse of \( R2? \)
   (b) (10) Find the conjugacy class of \( R3 \).
   (c) (5) Find \( (R3)^{201} \).
   (d) (−)(5) Is \( D_{10} \) Abelian? Give a short reason.

7. (25) Let \( R \) be a Euclidean Ring. Prove that every ideal of \( R \) is a principal ideal.

8. (20) In \( \mathbb{Z}[i] \) use the Euclidean Algorithm to find \( \gcd(3 + 5i, 5 + 3i) \). Show all work, including how you found the quotients.
9. (20) Let \( p \) be a prime in the integers. Suppose \( a + bi \in \mathbb{Z}[i] \) has \( a^2 + b^2 = p^3s \) for some integer \( s \). Show that \( a + bi \) cannot be a prime in \( \mathbb{Z}[i] \).

10. (20) Let \( p \) be prime and let \( G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^3} \) under addition.
   
   (a) (5) Describe the elements of \( G \). How many are there?
   
   (b) (10) Describe all those \( \alpha \in G \) with \( p\alpha = 0 \). How many such \( \alpha \) are there?
   
   (c) (5) Give all of the possible orders of the elements of \( G \).

11. (-)(10) In \( \mathbb{Z}[i] \) is \( 9 + 4i \in (2 + i)^2 \)? Give your reason!

**DO *ONE* OF PROBLEMS 12,13**

12. (35) (The object of this problem is to describe all the units \( \beta \in \mathbb{Z}[\sqrt{2}] \) with \( \beta > 1 \).) Let \( \alpha = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \) be a unit. Assume \( \alpha > 1 \).
   
   (a) (10) Show that \( -1 < a - b\sqrt{2} < +1 \).
   
   (b) (*) (10) Assume further \( 1 < \alpha < 2.5 \). Show that \( 0 < 2a < 3.5 \).
   
   (c) (-) (5) Show that \( 1 + \sqrt{2} \) is a unit. (As \( 1 + \sqrt{2} = 2.414\cdots < 2.5 \) it is therefore the smallest unit bigger than one.)
   
   (d) (*) (10) Let \( \beta \in \mathbb{Z}[\sqrt{2}] \) be a unit with \( \beta > 1 \). Prove that \( \beta = (1 + \sqrt{2})^s \) for some positive integer \( s \). (Hint: Consider that \( t \) with \( (1 + \sqrt{2})^t \leq \beta < (1 + \sqrt{2})^{t+1} \).)

13. (35) Set \( F = \mathbb{Z}_2[x]/(x^5 + x^2 + 1) \). Assume as a fact that \( x^5 + x^2 + 1 \in \mathbb{Z}_2[x] \) is irreducible.
   
   (a) (5) How many elements are in \( F \)?
   
   (b) (-) (5) Find \( (x^3 + x) + (x^3 + x + 1) \) in \( F \).
   
   (c) (5) Find \( (x^3 + x)(x^3 + x + 1) \) in \( F \).
   
   (d) (10) Find the multiplicative inverse of \( x \) in \( F \).
   
   (e) (10) Find \( x^{31000002} \) in \( F \). (Hint: There is a shortcut!)

Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest are can show.

The Dihedral Group $D_{10}$ is the group of symmetries of the regular 5-gon. We imagine the vertices of the regular 5-gon labelled 0, 1, 2, 3, 4 in counterclockwise direction.

The symmetries come in three forms:

1. $R_i, i = 1, 2, 3, 4$. This is a rotation by $i$ notches in the counterclockwise direction.

2. $F_i, i = 0, 1, 2, 3, 4$. This is a flip around point $i$.

3. The identity $e$.

A Table for the Dihedral Group $D_{10}$.

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