Introduction to Wavelets
Lecture Notes

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1 INTRODUCTION

Wavelets are special functions that can be used to represent other functions. The functions of interest to us are defined on the real domain, $\mathbb{R}^d$ for some $d \geq 1$ with complex range $\mathbb{C}$,

$$f : \mathbb{R}^d \to \mathbb{C}.$$

Often, the range of f can be restricted to $\mathbb{R}$. One dimensional functions ($d = 1$) are exemplified by sound waves; two dimensional function $f(x, y)$ may represent images. In order to have nice mathematical properties, we usually restrict the functions to some nice function spaces such as $L^1(\mathbb{R})$ or $L^2(\mathbb{R}^2)$. In the following, let L denote the class of functions under discussion (which may be taken $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ unless otherwise noted).

A set $B \subseteq L$ is called a **basis** for L if every function $f \in L$ can be expressed as a 'superposition' (linear combination) of functions in B,

$$f(x) = \sum_{i=0}^{\infty} c_i \psi_i(x), \quad c_i \in \mathbb{R}, \psi_i \in B.$$

In particular, B is called a **wavelet basis** when it is constructed in a special way. This idea of forming suppositions goes back to the French engineer Joseph Baptiste Fourier (1807) who asserted that any function $f(x)$ with a period 2π can be written as the superposition

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cos ix + b_i \sin ix).$$

Here, the basis functions are $\frac{1}{2}$, $\sin ix$ and $\cos ix$ for $i = 1, 2, \ldots$.

We view x as the 'time' (or space, depending on application) dimension of the function $f(x)$. Then each term $a_j \cos jx + b_j \sin jx$ ($j \geq 0$) in the above superposition can be viewed as a 'frequency' component of $f(x)$. In fact, a_j, b_j is easily seen to be equal the following integrals:

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \quad b_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx.$$

[Show this!] So a periodic function $f(x)$ may be represented *in the frequency domain* by the discrete set of coefficients $\{a_j, b_j : j \geq 0\}$. More generally, a function $f(x) \in L$ (not necessarily periodic) can be represented in the frequency domain by its **Fourier transform** which is defined (see NOTES) as follows:

$$F(f)(y) := \widehat{f}(y) = \int e^{-\mathbf{i}xy} f(x)dx, \qquad \mathbf{i} = \sqrt{-1}. \tag{1}$$

Note that even if f has real range, its transform $\widehat{f}$ will have complex range. It is easy to recover $f(x)$ from $\widehat{f}(y)$ using the **Fourier inversion formula**,

$$F(F(f))(x) = \widehat{\widehat{f}}(x) = 2\pi \cdot f(-x).$$

The study of this transform leads to the subject of Fourier or Harmonic analysis.

Despite the importance and success of this kind of 'frequency' analysis, the use of sines and cosines as basis is unsatisfactory for analysing functions $f(x)$ that are 'localized' in x (for instance a musical note $f(x)$ is localized in time). Clearly, sines and cosines are not localized in this sense. We want basis functions $\psi_i(x)$ that decay rapidly with $|x|$. One can introduce 'windowed Fourier transform' to analyze local signals. This is done by Dennis Gabor (1946) who introduced a family of functions based on the Gaussian curve. But wavelets turn out to give a better alternative because it fits in the general framework of superposition of functions. In retrospect, the first wavelet basis was already introduced by Alfred Haar (1909). The term "wavelet" was coined by Norman Ricker (1940) in seismology to describe the characteristic localized wave that moves outward from a seismic impulse. Morelet (1982) showed how these seismic wavelets can be modeled by the above functions of Gabor. Grossman and Morlet (1980) defined wavelets in quantum physics. Yves Meyer (1986) and Stephane Mallat (1988) developed these ideas into the theory of **multiresolution analysis** (MRA). In particular, Mallat introduce fast computational tools associated with MRA. Shortly after, the subject has exploded, and has found applications in many subjects.

There are many excellent books on Wavelets and popular wavelet articles on the web. Some references used in these notes are Koornwinder (ed.) [3], Hernández-Weiss [2] and Benectto-Frazier [1]. See the book of Stollnitz, DeRose and Salesin [4] for a gentle introduction, with applications to computer graphics.

SOME NOTATIONS. $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, reals and complex numbers, respectively. The support of a function $f(x)$ is the set $supp(f) := \{x : f(x) \neq 0\}$. The Kronecker delta function is $\delta(m) = 1$ if $m = 0$, and $\delta(m) = 0$ for $m \neq 0$. For any interval $[a, b]$, the characteristic function $\chi_{[a,b]}(x)$ is 1 iff $a \leq x \leq b$, and 0 otherwise. The integration $\int f(x)dx$ of a function $f \in L$ is assume to range over $\mathbb{R}$ unless otherwise stated. Likewise, in a summation $\sum_k c_k$ where k is an integer, it is assumed that k range over all integers $\mathbb{Z}$. The inner product of functions $f, g \in L$ is just $\langle f, g\rangle = \int f(x)\overline{g(x)}dx$. Note that functions in L have complex values in general, and $\overline{g(x)}$ here indicates complex conjugation: $\overline{x + \mathbf{i}y} = x - \mathbf{i}y$ where $\mathbf{i} = \sqrt{-1}$. Also, $|f|$ is defined to be $|\langle f, f\rangle|^{1/2}$. We say f and g are orthogonal if $\langle f, g\rangle = 0$. We should remember that functions in $L^2(\mathbb{R})$ are defined up to a set of measure zero. See Appendix A for more mathematical preliminaries.

2 HAAR WAVELETS

To have a concrete example as we introduce the abstract framework later, we describe Haar wavelets. Figure 1 shows a picture of a Haar scaling function $\phi(x)$ and its corresponding Haar wavelet $\psi(x)$:

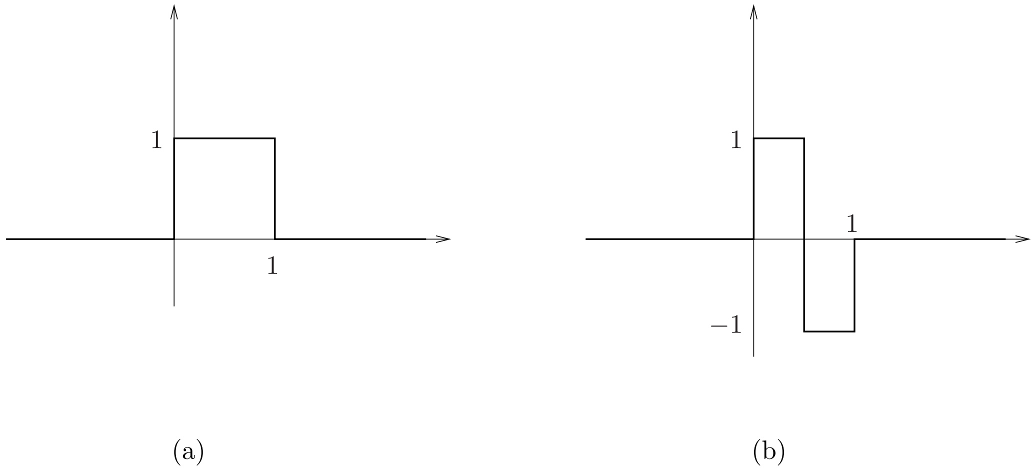

Figure 1: Haar Functions (a) Scaling $\phi(x)$ (b) Wavelet $\psi(x)$

Thus $\phi(x) = 1$ for $0 < x \leq 1$ and $\phi(x) = 0$ otherwise. Also, $\psi(x) = 1$ for $0 < x \leq 1$, $\phi(x) = -1$ for $-1 \leq x < 0$, and $\psi(x) = 0$ otherwise.

First consider the space $V_0 \subseteq L$ of all piecewise constant functions $f(x)$ with breakpoints at integer values of x. Clearly, $f(x)$ can be written as a superposition of integer translates of the Haar scaling function:

$$f(x) = \sum_{j=-\infty}^{\infty} a_j \phi(x - j).$$

Thus the set $\{\phi(x - j) : j \in \mathbb{Z}\}$ forms an orthonormal basis for V_0. In general, let V_i denote the set of all piecewise constant functions with breakpoints at x-values which are integer multiples 2^{-i}. For instance, the function in figure 2 is in V_1.

The functions

$$\phi_{i,j}(x) := \phi(2^i x - j)$$

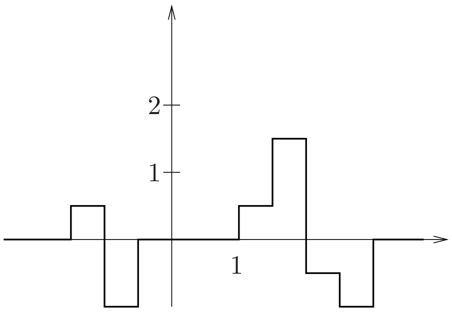

Figure 2: A function in V_2.

forms an orthogonal basis for V_i. It becomes orthonormal if we use $2^{i/2}\phi(2^i x - j)$ instead.

Now let us consider the relationship among these subspaces. Clearly, we have

$$V_i \subseteq V_{i+1}.$$

Consider the orthogonal complement W_i of V_i in V_{i+1},

$$V_{i+1} = V_i \oplus W_{i+1}. \tag{2}$$

By definition, a function $g(x) \in W_{i+1}$ iff $\langle g(x), f(x) \rangle = 0$ for all $f(x) \in V_i$. It is not hard to see that the functions

$$\psi_{i,j} := \psi(2^i x - j)$$

forms a basis for W_{i+1}.

EXERCISE: Decompose the function in figure 2 into a function in V_0 and a function in W_0.

EXERCISE: what are the breakpoints and support functions for $\phi_{i,j}$ and $\psi_{i,j}$?

Next, we can repeatedly expand the equation (2) to yield

$$\begin{aligned} V_i &= V_{i-1} \oplus W_i \\ &= V_{i-2} \oplus W_{i-1} \oplus W_i \\ &= \cdots \\ &= V_0 \oplus \left(\bigoplus_{j=1}^{i} W_j \right). \end{aligned}$$

In fact, we can continue this expansion for negative indices to yield

$$V_i = \bigoplus_{j=-\infty}^{i} W_j.$$

We note two other properties. Intuitively, it is clear that a function $f(x) \in L$ can be arbitrarily closely approximated by functions in V_i provided $i \to \infty$. This can be expressed by saying that the set

$$\bigcup_{i=0}^{\infty} V_i.$$

is dense in L. Second, note that

$$\bigcap_{i=0}^{\infty} V_{-i} = \{0\}$$

where 0 is only the identically zero function.

MNEMONICS: You might think of "W" as suggesting "wavelets" or the gentle curves associated with the sine wave. Note that the function $\psi(2x)$ is just like $\psi(x)$, but "oscillating" twice as fast (of course, any function will do in place of $\psi(x)$).

3 MULTIRESOLUTION ANALYSIS

A **multiresolution analysis** (MRA) for a space L is tower of subspaces

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq L \tag{3}$$

together with a **scaling function** (or "father wavelet") ϕ such that

1. (scaling) $f(x) \in V_i$ iff $f(2x) \in V_{i+1}$.
2. (basis) $\{\phi(x-j) : j \in \mathbb{Z}\}$ is an orthonormal basis for V_0.
3. (separation) $\bigcap_{i=-\infty}^{\infty} V_i = \{0\}$ where 0 is the identically zero function.
4. (density) $\bigcup_{i=-\infty}^{\infty} V_i$ is dense in L.

The above definition can be generalized in several ways. We can replace condition 2 by an apparently more general condition:

2*. (Riesz basis) $\{\phi(x-j) : j \in \mathbb{Z}\}$ is a Riesz basis.

But in fact, it can be shown that in the presense of the the other conditions, condition 2 and 2* are interchangeable. This gives us some flexibility in choosing ϕ for an MRA. See [3] (p. 21) for some basic facts about Riesz basis, and more generally the concept of a frame.

We can also work with functions $f(x)$ over $\mathbb{R}^d$ ($d = 1, 2, \ldots$). Then (following [Strichartz]) we can replace $\mathbb{Z}$ by a lattice $\Gamma \subseteq \mathbb{R}^d$. Then the scaling property can be replaced by

$$f(x) \in V_i \longleftrightarrow f(Mx) \in V_{i+1}$$

where M is a linear transformation of $\mathbb{R}^d$ satisfying $M\Gamma \subseteq \Gamma$ and all the eigenvalues λ of M satisfy $|\lambda| > 1$. In this case $|\det M|$ is an integer $m \geq 2$ and $\Gamma/M\Gamma$ is a group of order m. In the simplest case, $\Gamma = \mathbb{Z}^d$ is the unit lattice and $M = 2I$ with $m = 2^d$.

These conditions for MRA are also not independent: thus, [2] shows that condition 3 (separation) is implied by conditions 1 and 2*. Assuming that the Fourier transform of ϕ has the property that $|F(\phi)(y)|$ is continuous at $y = 0$. Then

$$\text{Condition 4 (density)}| \Leftrightarrow F(\phi)(0) = \widehat{\phi}(0) \neq 0 \Leftrightarrow |\widehat{\phi}(0)| = 1.$$

NOTATION: in the literature, there are two conventions for indexing the tower of subspaces in an MRA. Mallat uses the convention $V_i \subseteq V_{i-1}$ while our notation follows Daubechies. Let us write V^{-i} for V_i so that we have $V^i \subseteq V^{i-1}$, to get an indexing consistent with Mallat. The naturalness of each indexing scheme (V_i or V^i) can be explained this way. If you are more interested in the analysis of signals, then the Mallat convention makes more sense. That is, you begin with a signal $f = f^0 \in V^0$, and you can produce successively coarser approximations $f^1 \in V^1$, $f^2 \in V^2$, etc. Following Mallat, we call the superscript i in V^i the "scale" of f^i, and this is the inverse of the "resolution". If you want to do signal synthesis, then you begin with a signal $f = f_0 \in V_0$ and by adding more details, you can refine it to $f_1 \in V_1$, $f_2 \in V_2$, etc.

4 DILATION EQUATION

The scaling property

$$f(x) \in V_i \iff f(2x) \in V_{i+1}$$

and the fact that $\{\phi(t-j) : j \in \mathbb{Z}\}$ is an orthogonal basis for V_0 implies that

$$\{2^{i/2}\phi(2^i x - j) : j \in \mathbb{Z}\}$$

is an orthogonal basis for V_i. EXERCISE: show this.

In general, for any function $g(x)$ and $j, k \in \mathbb{Z}$, we define

$$g_{j,k}(x) := 2^{j/2} g(2^j x - k)$$

Then the above shows that $\{\phi_{j,k} : k \in \mathbb{Z}\}$ is an orthogonal basis for V_j.

Restricting attention to V_1, this means

$$\phi(x) = \sqrt{2} \sum_k c_k \phi(2x - k) \tag{4}$$

for some c_k's. This is called the **dilation equation**. We can of course obtain c_k as the projection of ϕ onto $\sqrt{2}\phi(2x-k)$,

$$c_k = \sqrt{2} \int \phi(x) \overline{\phi(2x-k)} dx.$$

Let us now derive several properties of the c_k's.

PROPERTY I, ORTHOGONALITY. For each $m \in \mathbb{Z}$,

$$\sum_j c_j c_{j-2m} = \delta(m).$$

In proof, use the fact that $\int \phi(x)\phi(x-m)dt = \delta(m)$, and substitute the dilation equation for $\phi(x)$ and for $\phi(x-m)$.

PROPERTY II, NORMALITY.

$$\sum_j |c_j|^2 = 1.$$

Proof sketch: use the fact that $\int \phi(x)\overline{\phi(x)}dx = 1$ and substitute the dilation equation for $\phi(x)$.

PROPERTY III, DILATION IN FOURIER DOMAIN: Dilation in the Fourier domain is expressed by

$$\widehat{\phi}(\omega) = H\left(\frac{\omega}{2}\right) \cdot \widehat{\phi}\left(\frac{\omega}{2}\right) \tag{5}$$

where

$$H(\omega) = \frac{1}{\sqrt{2}} \sum_k c_k e^{-\mathbf{i}\omega k} = \sum_k h_k e^{-\mathbf{i}\omega k}, \tag{6}$$

where we write h_k for $2^{-1/2} c_k$. In proof, we just take the Fourier transform of both sides of the dilation equation to give us

$$\widehat{\phi}(\omega) = \sum_k h_k e^{-\mathbf{i}k\omega} \widehat{\phi}(\omega/2).$$

The transformation of this infinite sum to the (5) actually require justification (for proof, see [3], p.54).

Iterating this, we obtain

$$\widehat{\phi}(\omega) = \widehat{\phi}(0) \prod_{i=0}^{\infty} H\left(\frac{\omega}{2^i}\right).$$

Note that

$$\widehat{\phi}(0) = \sqrt{2} \sum_k c_k \int \phi(2x-k)dx = 2^{-1/2} \sum_k c_k,$$

by the normalization assumption

$$\int \phi(t)dt = 1.$$

We may also assume $\widehat{\phi}(0) = 1$. Hence

$$\widehat{\phi}(\omega) = \prod_{i=0}^{\infty} H\left(\frac{\omega}{2^i}\right).$$

This infinite product is defined if $H(0) = 1$. Even if $H(0) \neq 1$, the infinite product may be well defined (because of fortuitious cancellation: see [Collela] in [1]) Note also that $H(\omega)$ is a 2π-periodic function.

For any function $g \in L^2(\mathbb{R})$, we may define a function $g^*(\xi)$ as follows

$$g^*(\xi) := \sum_k |\widehat{g}(\xi + 2\pi k)|^2 . \tag{7}$$

Thus g^* is roughly speaking, the 2π-periodization of $|\widehat{g}(xi)|^2$, where $\widehat{g}$ is the Fourier transform of g.

Lemma 1 (Basic Fourier Tool) *Let $g \in L^2(\mathbb{R})$. Then the following two statements are equivalent:*
(i) $\{g(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system.
(ii) The 2π-periodization of $|\widehat{g}(xi)|^2$ is equal to 1 almost everywhere:

$$\sum_k |\widehat{g}(\xi + 2\pi k)|^2 = 1, \qquad \text{(for almost all } \xi \in \mathbb{R}). \tag{8}$$

Proof.

$$\begin{aligned}
\delta(k) &= \int g(x)\overline{g(x-k)}dx & (9)\\
&= \frac{1}{2\pi}\int \widehat{g}(\xi)\widehat{g(\xi - k)}d\xi \quad (Plancheral) & (10)\\
&= \frac{1}{2\pi}\int |\widehat{g}(\xi)|^2 e^{\mathbf{i}k\xi}d\xi & (11)\\
&= \frac{1}{2\pi}\sum_\ell \int_{2\ell\pi}^{2(\ell+1)\pi} |\widehat{g}(\mu + 2\pi\ell)|^2 e^{\mathbf{i}k\mu}d\mu & (12)\\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{\mathbf{i}k\mu}\left(\sum_\ell |\widehat{g}(\mu + 2\pi\ell)|^2\right) d\mu & (13)\\
&. & (14)
\end{aligned}$$

This shows that $\{\delta(k) : k \in \mathbb{Z}\}$ are the Fourier coefficients of the $g^*(\xi)$, and the lemma follows. **Q.E.D.**

Using this, we derive another property:

PROPERTY IV.

$$\left|H\left(\frac{\omega}{2}\right)\right|^2 \left|H\left(\frac{\omega}{2}\right) + \pi)\right|^2 = 1$$

In proof,...

Other remarks:
1. $H(\omega)$ is a trigonometric polynomial if the c_k's are zero for $|k| > N$ (some N).
2. The case where $c_k = 0$ for $|k| > N$ can be shown to lead to a unique compact solution $\phi(t)$ to the dilation equation. [Show]
3. The dilation equation can be viewed as a fixed point operator, and it can be used to compute $\phi(t)$. It is easy to see from this iteration that the solution $\phi(t)$ is supported in the interval $[N, -N]$.
4. It is instructive to illustrate all the above equations with the case of Haar functions where $c_0 = c_1 = \frac{1}{\sqrt{2}}$.

APPENDIX A: MATHEMATICAL BACKGROUND

The class $L^1(\mathbb{R})$ of **integrable functions** comprise those $f : \mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |f(t)|dt < \infty$.

Two functions f, g in this class are considered equal if $\int_{-\infty}^{\infty} |f(t) - g(t)|dt = 0$. We then say that f, g are **almost everywhere equal**.

APPENDIX B: NOTES

Definitions of Fourier Transform. Definitions of the Fourier transform vary in the literature: there is a choice of constant factors in front of the integral and/or in the exponent inside the integral. In particular, $\mathbf{i}$ may be replaced by $-\mathbf{i}$. For instance, if we define the Fourier transform to be

$$F(f)(y) = \int e^{2\pi \mathbf{i} x \cdot y} f(x) dx$$

and its inverse transform to be

$$F^{-1}(f)(x) = \int e^{-2\pi \mathbf{i} x \cdot y} f(y) dy,$$

then the Fourier inversion formula says that

$$F^{-1}(F(f)) = f.$$

References

[1] John J. Benedetto and Michael W. Frazier. *Wavelets: Mathematics and Applications*. CRC Press, Boca Raton, 1994.

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[4] Eric J. Stollnitz, Tony D. DeRose, and David H. Salesin. *Wavelets for Computer Graphics.* Morgan Kaufmann Publishers, Inc, San Francisco, 1996.