On invariants of finite dimensional pointed Hopf algebras *

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An action of finite dimensional Hopf algebra H on arbitrary commutative algebra A is considered. The example of such an action is constructed refuting the hypothesis of Susan Montgomery that every extension A/A^H is integral under above conditions (where A^H is subalgebra of H-invariants in A). Next it is proved that this hypothesis is true in some partial cases of pointed Hopf algebras (if H is commutative or $\operatorname{char} k = p > 0$).

1 Introduction

Throughout this paper H is a finite dimensional Hopf algebra over a field k, and A is a associative k-algebra.

Definition 1.1 *It is said that H acts on A, if A is left H-module and for any $h \in H, \quad a, b \in A$*

$$h(ab) = \sum_h (h_{(1)}a)(h_{(2)}b), \quad h1 = \varepsilon(h),$$

where $\varepsilon : H \to k$ - counit and Δ - comultiplication:

$$\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)} \in H \otimes H.$$

Often A is called H-module algebra.

Definition 1.2 *The invariants of H in A is the set A^H of those $a \in A$, that $ha = \varepsilon(h)a$ for each $h \in H$.*

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Straightforward computations show, that A^H is subalgebra of A. We refer reader to [5], [6] for the basic information concerning Hopf algebras and their actions on associative algebras.

As a generalization of the well-known fact for group actions the following question was raised in [5] (Question 4.2.6.)

Question 1.3 *If A is a commutative k-algebra and H any finite dimensional Hopf algebra such that A is H-module algebra, is A integral over A^H ?*

If A is affine algebra, then Artin-Tate lemma ensures that A^H is also affine.

Some positive answers to question 1.3 are known.

Theorem 1.4 ([2]) *Let H be a finite dimensional cocommutative Hopf algebra and let A be a commutative H-module algebra. Then A is integral extension of A^H.*

The main result of the paper is stated in the theorem 2.6. It gives positive answer to Question 1.3 in some partial cases. Let H be pointed Hopf algebra and let A be affine H-module algebra; if one of two conditions is satisfied, then A is integral over A^H:

1. H is commutative as an algebra,
2. $\operatorname{char} k = p > 0$,

We recall that Hopf algebra H is called *pointed* if every simple subcoalgebra of H is one-dimensional; pointed Hopf algebra is called *connected* if it has only one simple subcoalgebra (one-dimensional). The examples of pointed Hopf algebras are given by group algebras, universal enveloping algebras. In fact, if G - group, then the only simple subcoalgebras of kG are those of the form kg, $g \in G$. At the same time universal enveloping algebras are examples of connected Hopf algebras: the only simple subcoalgebra of universal enveloping algebra is $k1_H$.

Another important example of pointed Hopf algebras is represented by series of Hopf algebras A_N, where $N \geq 2$ – integer number, considered in [3] (see also section 3 of this paper). Note that with $N = 2, \operatorname{char} k \neq 2$ algebra A_2 (sometimes called H_4) is the only Hopf algebra of minimal dimension which is neither cocommutative nor commutative ([5], example 1.5.6).

In spite of numerous partial positive results it turned out that the hypothesis 1.3 isn't true in general, even for pointed Hopf algebras. The counterexamples are constructed in section 3 for the series of pointed Hopf algebras $A_N, N \geq 2$, mentioned at the previous paragraphs [1].

[1] After the work had been done author became aware through e-mail by Susan Montgomery, that the counterexample to the hypothesis 1.3 was independently constructed by Zhu Shenglin. Also he obtained some positive results, solving the problem 1.3. His paper "Integrality of module algebras over its invariants" should have appeared in J.Algebra in 1996.

2 The main theorem

The proof of the theorem is based on the properties of a coradical filtration of arbitrary coalgebra. We recall the basic facts.

Definition 2.1 *The coradical C_0 of coalgebra C is the (direct) sum of all simple subcoalgebras in C. Further by induction for each $n \geq 1$ define*

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$$

Theorem 2.2 ([5], Theorem 5.2.2) *$\{C_n\}_{n\geq 0}$ is a family of subcoalgebras with the following properties:*

1. $C_{n-1} \subseteq C_n, \quad C = \bigcup\limits_{n\geq 0} C_n,$

2. $\Delta C_n \subseteq \sum\limits_{i=0}^{n} C_i \otimes C_{n-i}.$

Reader may find more detailed description of the coradical filtration in [6], chapter 9.

Let H be a pointed finite dimensional Hopf algebra over field k. Let $G = G(H)$ denote the set of *grouplike* elements of H, i.e.,

$$G = \{ \quad g \in H \setminus 0 \quad | \quad \Delta g = g \otimes g \quad \}.$$

It is known that the elements of G are linearly independent, G is a group under multiplication arisen from the multiplication in H, and subalgebra generated by G is a group Hopf algebra kG. Also kG is a coradical of H.

By lemma 5.2.8 [5], the coradical filtration $\{H_n\}$ of Hopf algebra H is a Hopf filtration of Hopf algebra, i.e., $\Delta H_n \subseteq \sum_{i=0}^{n} H_i \otimes H_{n-i}, \quad H_n H_m \subseteq H_{n+m}$, $SH_n \subseteq H_n$ for all $n, m \geq 0$, if and only if H_0 is a sub-Hopf algebra of H. If H – pointed finite dimensional Hopf algebra, then this condition is obviously satisfied. Moreover, the coradical filtration is finite.

By theorem 5.4.1 from [5] (see also [7], [4] for reference), the coradical filtration $\{H_n\}$ of pointed Hopf algebra H has additional properties. If $x \in H_m$, $m \geq 1$, then

$$x = \sum_{g,h \in G(H)} c_{g,h}, \tag{1}$$

where

$$\Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w, \qquad w \in H_{m-1} \otimes H_{m-1}. \tag{2}$$

Note that if $a, b, g, h \in G$ and $c = ac_{g,h}b$, then by (2)

$$\Delta(c) = c \otimes agb + ahb \otimes c + w', \qquad w' \in H_{n-1} \otimes H_{n-1}.$$

Define $H^+ = \ker \varepsilon, \quad H_r^+ = H_r \cap H^+$. Let A^G denote the subalgebra of G-invariants in A ($A^H \subseteq A^G$). Extension A/A^G is integral by the Noether's theorem for a group actions (H – finite dimensional, therefore G – finite group).

Before we start to prove the main theorem we are going to obtain a few auxiliary results.

Let I denote the ideal in H generated by the elements of form $g-1, \quad g \in G$.

Proposition 2.3 *I is a Hopf ideal in H.*

Proof. If $g \in G$, then

$$\Delta(g-1) = g \otimes g - 1 \otimes 1 = (g-1) \otimes g + 1 \otimes (g-1) \in I \otimes H + H \otimes I.$$

$S(I) \subseteq I$, because $S(g-1) = g^{-1} - 1$ and S is an anti-homomorphism. This yields the proposition. □

Proposition 2.4 *If J – Hopf ideal in H, then H/J – pointed Hopf algebra. Moreover, the natural epimorphism of Hopf algebras $\pi : H \to H/J$ induces the epimorphism of groups of grouplike elements $G(H) \to G(H/J)$.*

Proof. This statement is a direct consequence of corollary 5.3.5 from [5]. □

Theorem 2.5 *Let one of two following conditions be satisfied:*

1. $\operatorname{char} k = p > 0$.
2. *H – connected and commutative;*

Then there exists the chain of subalgebras $A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_n$ with the following properties:

1. *each extension $A_i \supseteq A_{i+1}$ is integral;*
2. *if $x \in H_i^+$ then $x(A_i) = 0$.*

Proof. To construct this chain we start with defining $A_0 = A^G$. Both of the necessary conditions are satisfied. Let the chain

$$A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_r, \quad r \geq 0,$$

be already constructed and let $x \in H_{r+1}^+$. By (1) we may assume that $x = c_{g,h}$, where $g, h \in G$. Then

$$\Delta(x) = x \otimes g + h \otimes x + \sum u_j \otimes v_j, \tag{3}$$

where $v_j, u_j \in H_r$. Moreover, it was shown in [1], that $u_j, v_j \in H_r^+$.

If $\operatorname{char} k = p > 0$, then we define $A_{m+1} = A_m^p$. Really, by (3)

$$x(a^p) = h(a)^{p-1}x(a) + h(a)^{p-2}x(a)g(a) + \cdots + x(a)g(a)^{p-1} = pa^{p-1}x(a) = 0.$$

Suppose H – connected, commutative Hopf algebra and $\operatorname{char} k = 0$, i.e., $g = h = 1$. By (3),

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum u_j \otimes v_j,$$

where $u_j, v_j \in H_r^+$. Under these assumptions we can fully apply the technique of [1] and show that $x(A_r) = 0$. So in this case we define $A_{r+1} = A_r = \cdots = A^G$. □

Now we have come to the main

Theorem 2.6 *Let A be an affine H-module algebra, H – finite dimensional pointed Hopf algebra and one of two conditions is satisfied:*

1. $\operatorname{char} k = p > 0$*;*

2. *H – commutative;*

Then extension A/A^H is integral.

Proof. Assume that H is commutative, then A^G is an H-module algebra. It is sufficient to show that A^G is stable under H-action. In fact, for each $x \in H, \quad a \in A$,

$$gx(a) = xg(a) = x(a),$$

i.e., $x(a) \in A^G$. Let I denote the ideal in H generated by the elements of form $g - 1$ $(g \in G)$. By proposition 2.3 I is a Hopf ideal. Obviously, it acts as zero on A^G. Hopf algebra H/I is pointed by proposition 2.4, moreover, it is connected. Thus the second case of this theorem is reduced to the consideration of an action of connected commutative Hopf algebra H/I on the algebra A^G. Now we apply theorem 2.5 to the both cases. Let

$$A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_n$$

be constructed chain of subalgebras. By condition 1) of theorem 2.5, extension A/A_n is integral; by condition 2) $A_n \subseteq A^H$. □

Note that if $\operatorname{char} k = p > 0$, then $(A^G)^{p^{\dim H}} \subseteq A^H$. If H is commutative and $\operatorname{char} k = 0$, then $A^H = A^G$.

3 Counterexample to hypothesis

Example 3.1 Hopf algebra H may be any one from the series $A_N, \quad N \geq 2$. Algebra A_N is generated by the elements g, x with the relations

$$g^N = 1, \quad x^N = 0, \quad xg = \xi gx, \tag{4}$$

where $\xi \in k$ – prime root of unity of degree N. A Hopf algebra structure on A_N is given as follows:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{N-1} = g^{-1},$$

$$\Delta(x) = g \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -g^{N-1}x.$$

We demand that $\operatorname{char} k = 0$. Algebra A_N is pointed,

$$G(A_N) = \{1, g, g^2, \ldots, g^{N-1}\},$$

it is non-commutative and non-cocommutative.

Let A be the commutative algebra generated by the elements y, z with the relation $z^2 = 0$. Define the action of A_N on A:

$$g(y^n) = y^n, \quad g(y^n z) = \xi^{-1} y^n z, \quad x(y^n) = ny^{n-1}z, \quad x(y^n z) = 0.$$

Straightforward computations show the correctness of this action, i.e., $A_N(I) \subseteq I$, where I is the ideal of free algebra $k < y, z >$ generated by the elements $yz - zy, z^2$. We check that relations (4) in H are satisfied:

$$\xi gx(y^n) = \xi\xi^{-1} ny^{n-1}z = ny^{n-1}z = xg(y^n),$$

$$\xi gx(y^n z) = 0 = xg(y^n z),$$

$$g^N(y^n) = y^n, \quad g^N(y^n z) = \xi^{-N} y^n z = y^n z,$$

$$x^2(y^n) = nx(y^{n-1}z) = 0, \quad x^2(y^n z) = x(0) = 0,$$

i.e., $x^N(a) = x^2(a) = 0$ for any $a \in A$.

Obviously $A^G = k[y]$ and $A^H = k[y] \bigcap \ker x = k$. But extension A/A^H is not integral, because A is not finite k-module ($\dim_k A = \infty$).

4 Conclusion

As it was shown in the example 3.1, the hypothesis 1.3 is not true in general. Nevertheless all known examples of Hopf algebra action show that if A – affine, then A^H is also affine algebra, although extension A/A^H is not always integral. So we ask

Question 4.1 *A finite dimensional Hopf algebra H acts on a commutative affine algebra A. Is A^H affine?*

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Revised and completed compilation of [1] and this paper is also available at the Quantum Algebra and Topology e-Print archive:
`http://xxx.lanl.gov/abs/q-alg/9611017`

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