Actions of Hopf algebras on noncommutative algebras *

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1 Introduction

Throughout this paper H is a finite-dimensional Hopf algebra over a field k, and A is a associative k-algebra. Unless it is stated additionally, all tensor products are over k.

Definition 1.1 *It is said that H acts on A, if A is left H-module and for any $h \in H, \quad a, b \in A$*

$$h(ab) = \sum_h (h_{(1)}a)(h_{(2)}b), \quad h1 = \varepsilon(h),$$

where $\varepsilon : H \to k$ - counit and Δ - comultiplication:

$$\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)} \in H \otimes H.$$

Often A is called H-module algebra. We refer reader to [11, 6] for the basic information concerning Hopf algebras and their actions on associative algebras.

Definition 1.2 *The invariants of H in A is the set A^H of those $a \in A$, that $ha = \varepsilon(h)a$ for each $h \in H$.*

Straightforward computations show, that A^H is subalgebra of A.

The notion of action of Hopf algebra on associative algebra generalize the notion of automorphism and derivation of associative algebra. As a generalization of the well-known fact for group actions the following question was raised in [6] (Question 4.2.6.)

Question 1.3 *If A is a commutative k-algebra and H any finite-dimensional Hopf algebra such that A is H-module algebra, is A integral over A^H ?*

Some positive answers to question 1.3 are known.

Theorem 1.4 ([5]) *Let H be a cocommutative Hopf algebra and A be a commutative H-module algebra. Then A is integral extension of A^H.*

Theorem 1.5 ([1]) *Finite-dimensional pointed Hopf algebra H acts on affine integral domain A, then extension A/A^H is integral.*

*The work was presented at the ICM'98 as a short communication and is based on the paper A.A. Totok, Deistviya algebr Hopfa (Actions of Hopf algebras), *Matematichesky Sbornik* **189**, No. 1 (1998), 149 – 160 (in russian).

Theorem 1.6 **([14])** *Let H be finite-dimensional involutory ($S^2 = I$) Hopf algebra and* $\operatorname{char} k \nmid \dim H$*, then A is integral over A^H.*

Theorem 1.7 **([13])** *Finite-dimensional pointed Hopf algebra H acts on commutative algebra A and one of the following conditions is satisfied:*

1. $\operatorname{char} k = p > 0$,
2. *H is commutative.*

Then extension A/A^H is integral.

We recall that Hopf algebra H is called *pointed* if every simple subcoalgebra of H is one-dimensional. The examples of pointed Hopf algebras are given by group algebras and universal enveloping algebras. In spite of numerous partial positive results it turned out that the hypothesis in question 1.3 isn't true in general even for pointed Hopf algebras. The counterexamples are constructed in [14] and [13].

Questions similar to the question 1.3 may arise for noncommutative algebras (see [6] § 4.3). Let $B \subset A$ – be an extension of noncommutative algebras. If $a \in A$ then (B, a)-monomial is a product in A whose each factor either is a or $\in B$; the degree of such monomial is the number of a's in it. If $a_1, \dots, a_m \in A$ then $(B, a_1, \dots, a_m)$-monomial is a product in A, whose each factor is either any a_i or an element from B. The degree of such monomial is the number of a_i in it.

Definition 1.8 *Algebra A is called Schelter-integral over B, if for any $a \in A$ there exists integer n that $a^n = \phi$ where ϕ is a sum of (B, a)-monomials of degree less than n.*

Definition 1.9 *Algebra A is called fully-integral over B of degree n, if for any $a_1, \dots, a_n \in A$ $\quad a_1 a_2 \dots a_{n-1} a_n = \psi$, where ψ is a sum of $(B, a_1, \dots, a_n)$-monomials of degree less than n.*

It is obvious that if extension $B \subset A$ is fully integral then it is also Schelter-integral of bounded degree. The following result similar to the commutative case was obtained in [9]:

Theorem 1.10 *Finite group G acts by automorphisms on noncommutative algebra A and* $\operatorname{char} k \nmid |G|$*, then A is fully integral over A^G of degree $\leq 2^{2^{|G|+1}}$*

The example was constructed in [8] with $\operatorname{char} k$ dividing $|G|$ and extension A/A^G even not Schelter-integral. Unlike the commutative case there are a lot of open questions in the noncommutative case (see [6] § 4.3).

We consider Hopf algebra actions on noncommutative algebras. With the help of the argument of extension of the base field to its algebraic closure, consideration of the class of Hopf algebras with cocommutative coradical is reduced to the class of pointed Hopf algebras.

In section 2 we consider restriction of an action of Hopf algebra H on the center $\mathrm{Z}(A)$ of an algebra A, obtaining some generalizations of theorems 1.4, 1.5, 1.7 for noncommutative algebras integral over their centers. Moreover, the condition of algebra A being integral domain in the theorem 1.5 is substituted by more weaker condition of absence of nilpotent

elements in A. As example from [13] shows we can not make weaker this condition even for commutative algebra A.

In section 3 we consider an action of commutative Hopf algebra H on arbitrary algebra A. With the technique developed in [1] and [13] some structure theorems for commutative Hopf algebras are proved: Let H be a commutative finite-dimensional Hopf algebra and $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H$. If the coradical H_0 is a sub-Hopf algebra of H or cocommutative, then H is cosemisimple, i.e. $H = H_0$. In particular, a commutative pointed Hopf algebra with $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H$ is a group Hopf algebra. Also the example is constructed showing that restrictions on $\operatorname{char} k$ are necessary.

2 Pointed Hopf algebras

The technique we use is based on the properties of the coradical filtration of arbitrary coalgebra. We recall the basic facts.

Definition 2.1 *The coradical C_0 of coalgebra C is the (direct) sum of all simple subcoalgebras in C. Further by induction for each $n \geq 1$ define*

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$$

Theorem 2.2 ([6], Theorem 5.2.2) *$\{C_n\}_{n\geq 0}$ is a family of subcoalgebras with the following properties:*

1. $C_{n-1} \subseteq C_n, \quad C = \bigcup\limits_{n\geq 0} C_n,$
2. $\Delta C_n \subseteq \sum\limits_{i=0}^{n} C_i \otimes C_{n-i}.$

More detailed description of the coradical filtration may be found in [11] chapter 9.

By theorem 5.4.1 from [6] (see also [12]), the coradical filtration $\{H_n\}$ of pointed Hopf algebra H has additional properties. If $x \in H_m, m \geq 1$, then

$$x = \sum_{g,h\in G(H)} x_{g,h}, \tag{1}$$

where $G = G(H)$ is the set of grouplike elements in H and

$$\Delta(x_{g,h}) = x_{g,h} \otimes g + h \otimes x_{g,h} + w, \qquad w \in H_{m-1} \otimes H_{m-1}. \tag{2}$$

Define $H^+ = \ker \varepsilon, H_r^+ = H_r \cap H^+$. If $x \in \ker \varepsilon$, then $w \in H_{m-1}^+ \otimes H_{m-1}^+$ [1].

If H pointed finite-dimensional Hopf algebra then $H_0 = kG(H)$ is a group sub-Hopf algebra and the coradical filtration $\{H_i\}_{i=0}^n$ of Hopf algebra H is finite.

Theorem 2.3 *Let pointed Hopf algebra H act on noncommutative algebra A and one of the following conditions be satisfied:*

1. $\operatorname{char} k = p > 0,$

2. $\operatorname{char} k = 0$, *algebra A does not contain nilpotent elements,* $\mathrm{Z}(A)$ *(the center of A) is affine algebra.*

Then $\mathrm{Z}(A)/\mathrm{Z}(A)^H$ *is integral, where* $\mathrm{Z}(A)^H = \mathrm{Z}(A) \cap A^H$.

Proof. The point is that it is not always true that $H(\mathrm{Z}(A)) \subseteq \mathrm{Z}(A)$ for noncommutative algebra A, so we can not apply known theorems from the commutative case.

As in [1, 13] we prove that there exists the chain of subalgebras $\mathrm{Z}(A) = Z_{-1} \supseteq Z_0 \supseteq \ldots \supseteq Z_n$ in $\mathrm{Z}(A)$ such that for each i Z_{i-1}/Z_i is integral extension and if $x \in H_i^+$ then $x(Z_i) = 0$. This ensures that $\mathrm{Z}(A)/Z_n$ is integral extension, $\mathrm{Z}(A)^H \subseteq Z_n$ and thus $\mathrm{Z}(A)/\mathrm{Z}(A)^H$ is also integral.

Define $Z_0 = \mathrm{Z}(A)^G$. Extension $\mathrm{Z}(A)/Z_0$ is integral because $G(\mathrm{Z}(A)) \subseteq \mathrm{Z}(A)$ and G is a finite group of automorphisms of an algebra A. Let the chain $\mathrm{Z}(A) = Z_{-1} \supseteq Z_0 \supseteq \ldots \supseteq Z_m$ be constructed and $x \in H_{m+1}^+$. By (1) we may assume that $x = x_{g,h}$.

If $\operatorname{char} k = p > 0$ then define $Z_{m+1} = Z_m^p$. By (2)

$$\begin{aligned} x(z^p) = h(z)^{p-1}x(z) + h(z)^{p-2}x(z)g(z) + \ldots + h(z)x(z)g(z)^{p-2} + x(z)g(z)^{p-1} = \\ = z^{p-1}x(z) + z^{p-2}x(z)z + \ldots + zx(z)z^{p-2} + x(z)z^{p-1} = pz^{p-1}x(z) = 0, \end{aligned}$$

because $z \in \mathrm{Z}(A)^G$.

If $\operatorname{char} k = 0$, algebra $\mathrm{Z}(A)$ is affine, A does not contain nilpotent elements and $\mathrm{Z}(A)/Z_m$ is integral extension then by Artin-Tate lemma Z_m is affine algebra and by normalization lemma [3] chapter 5 §3, Z_m has subalgebra of polynomials $k[T_1, \ldots, T_s]$ and extension $Z_m/k[T_1, \ldots, T_s]$ is integral. Define $Z_{m+1} = k[T_1, \ldots, T_s]$ and prove that $x(Z_{m+1}) = 0$. If

$$\Delta(x) = x \otimes g + h \otimes x + w, \quad w \in H_m^+ \otimes H_m^+$$

then for element $y = xg^{-1}$ it is true that

$$\Delta(y) = y \otimes 1 + h' \otimes y + w', \quad w' \in H_m^+ \otimes H_m^+,$$

where $h' = hg^{-1}$. Thus for each $z \in Z_m$, $a \in A$

$$y(za) = y(z)a + h'(z)y(a) = y(z)a + zy(a), \tag{3}$$

i.e. $y : Z_m \to A$ is a derivation. Then we can apply the technique of [1]. For each $f \in k[T_1, \ldots, T_s]$ by (3)

$$y(f) = \sum_{i=1}^{s} \frac{\partial f}{\partial T_i} a_i, \quad a_i = y(T_i) \in A, \quad \frac{\partial f}{\partial T_i} \in \mathrm{Z}(A),$$

further

$$y^2(f) = \sum_{i,j=1}^{s} \frac{\partial^2 f}{\partial T_i \partial T_j} a_j a_i + \sum_{i=1}^{s} \frac{\partial f}{\partial T_i} y(a_i),$$

$$y^3(f) = \sum_{i,j,k=1}^{s} \frac{\partial^3 f}{\partial T_i \partial T_j \partial T_k} a_k a_j a_i \quad + \sum_{i,j=1}^{s} \frac{\partial^2 f}{\partial T_i \partial T_j} (y(a_j a_i) + a_j y(a_i)) + \sum_{i=1}^{s} \frac{\partial f}{\partial T_i} y^2(a_i).$$

In the same way for each integer $q \geq 1$

$$y^q(f) = \sum_{i_1,\ldots,i_q=1}^{s} \frac{\partial^q f}{\partial T_{i_1} \partial T_{i_2} \ldots \partial T_{i_{q-1}} \partial T_{i_q}} a_{i_q} a_{i_{q-1}} \ldots a_{i_2} a_{i_1} + \\ + \sum_{\substack{1 \leq j_1+\ldots+j_s=\ell<q \\ j_k \geq 0}} \frac{\partial^\ell f}{\partial T_1^{j_1} \ldots \partial T_s^{j_s}} \Psi_{j_1,\ldots,j_s}, \tag{4}$$

where $\Psi_{j_1,\ldots,j_s}$ is an algebraic expression not dependent on f and containing a_i and y in it. Since Hopf algebra H is finite-dimensional, then there exists such integer d that

$$y^d = \sum_{i=1}^{d-1} \beta_i y^i, \quad \beta_i \in k. \tag{5}$$

Therefore by (4) and (5) for each $f \in Z_{m+1}$ and any $i = 1, \ldots, s$ we have

$$\frac{\partial^d f}{\partial T_i^d} a_i^d + \sum_{\ell=1}^{d-1} \frac{\partial^\ell f}{\partial T_i^\ell} \Phi_\ell + \Lambda = 0, \tag{6}$$

where Φ_ℓ does not depend on f and Λ is the sum of all summands from (4) and (5) containing at least one derivation by the indeterminate different from T_i. Substituting in (6) successively $T_i, T_i^2, \ldots, T_i^d$ and using the fact that $\operatorname{char} k = 0$ we get that $\Phi_1 = \Phi_2 = \ldots = \Phi_{d-1} = a_i^d = 0$. As algebra A does not contain nilpotent elements, then $a_i = 0$ and $y(Z_{m+1}) = xg^{-1}(Z_{m+1}) = x(Z_{m+1}) = 0$. $\square$

Remark. As example from [13] shows, we can not make weaker the condition of absence of nilpotent elements in algebra A.

Next reasoning shows that actually we can consider much more wider class of Hopf algebras then pointed Hopf algebras – the class of Hopf algebras with cocommutative coradical. Let K be any extension of the base field k, define $\overline{A} = A \otimes_k K, \overline{H} = H \otimes_k K$, then following is true [2] Lemma 1:

1. $\overline{H}$ is a Hopf algebra over field K:
   $\overline{\Delta}(h \otimes t) = \sum_h (h_{(1)} \otimes t) \otimes_K (h_{(2)} \otimes 1)$,
   $\overline{\epsilon}(h \otimes t) = \epsilon(h)t$,
   $\overline{S}(h \otimes t) = S(h) \otimes t$;

2. $\overline{A}$ – $\overline{H}$-module algebra over K:
   $(h \otimes t)(a \otimes m) = h(a) \otimes tm$;

3. if B is a subalgebra of A then $\overline{B}^{\overline{H}} = \overline{B^H} = B^H \otimes K$, where $B^H = B \cap A^H$.

Now let K be algebraic closure of the base field k and H have cocommutative coradical H_0. As $\left(\overline{H}\right)_0 \subseteq \overline{(H_0)} = H_0 \otimes K$, then $\left(\overline{H}\right)_0$ is cocommutative and thus is a group Hopf algebra, because field K is algebraically closed [11] Lemma 8.0.1, i.e. $\overline{H}$ is a pointed Hopf algebra.

It is easy to show that $\mathrm{Z}(\overline{A}) = \overline{\mathrm{Z}(A)}$. Really, it is obvious that $\mathrm{Z}(\overline{A}) \supseteq \overline{\mathrm{Z}(A)}$. Backward, if $\sum_i a_i \otimes e_i \in \mathrm{Z}(\overline{A})$, where $a_i \in A$, $e_i \in K$ and $\{e_i\}$ are linearly independent over k then for each $b \in A$

$$\sum_i ba_i \otimes e_i = (b \otimes 1)(\sum_i a_i \otimes e_i) = (\sum_i a_i \otimes e_i)(b \otimes 1) = \sum_i a_i b \otimes e_i,$$

therefore $ba_i = a_i b$, i.e. $a_i \in \mathrm{Z}(A)$.

Commutative algebra A is integral over subalgebra B if and only if $\overline{A}$ is integral over $\overline{B}$. In noncommutative case it is true only in one direction: if $\overline{A}$ is fully-integral (resp. Schelter-integral) over $\overline{B}$ then A is fully-integral (resp. Schelter-integral) over B [2] Proposition 2.

Unfortunately, the property of absence of nilpotent elements are not inherited with the extension of the base field. Even a division algebra can "get" nilpotent elements under such operation (real quaternion algebra $\mathrm{R}(-1,-1)$ becomes $\mathrm{M}_2(\mathrm{C})$ with the extension of the field R to C).

Next lemma will be useful:

Lemma 2.4 ([4], Proposition 4) *If H is cocommutative then $H(\mathrm{Z}(A)) \subseteq \mathrm{Z}(A)$ for each H-module algebra A.*

Theorem 2.5 *Let Hopf algebra H act on noncommutative algebra A and A be integral over its center $\mathrm{Z}(A)$ (for example, if A is a finite $\mathrm{Z}(A)$-module), then A is integral over $\mathrm{Z}(A)^H$ in the following cases:*

1. *the coradical H_0 of H is cocommutative and* $\operatorname{char} k = p > 0$*,*
2. *H is pointed, A does not contain nilpotent elements, algebra $\mathrm{Z}(A)$ is affine and* $\operatorname{char} k = 0$*,*
3. *H is cocommutative.*

Proof. If H is cocommutative then $\mathrm{Z}(A)$ is an H-module algebra by lemma 2.4 and $\mathrm{Z}(A)/\,\mathrm{Z}(A)^H$ is integral by theorem 1.4. In case 1 use the argument of extension of the base field, so $\mathrm{Z}(A) \otimes K = \overline{\mathrm{Z}(A)} = \mathrm{Z}(\overline{A})$ is integral over $\mathrm{Z}(\overline{A})^{\overline{H}} = \overline{\mathrm{Z}(A)}^{\overline{H}} = \overline{\mathrm{Z}(A)^H} = \mathrm{Z}(A)^H \otimes K$ by theorem 2.3 ($\overline{H}$ is pointed, $\operatorname{char} K = \operatorname{char} k = p > 0$), therefore $\mathrm{Z}(A)$ is integral over $\mathrm{Z}(A)^H$. By the same theorem it is true in case 2. If A is integral over $\mathrm{Z}(A)$ and $\mathrm{Z}(A)$ is integral over $\mathrm{Z}(A)^H$ then extension $A/\,\mathrm{Z}(A)^H$ is also integral. $\square$

3 Commutative Hopf algebras

We consider commutative Hopf algebras. To start we prove important

Theorem 3.1 *Let commutative Hopf algebra H with cocommutative coradical H_0 act on arbitrary associative algebra A and either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > \dim H$*. Then A is fully-integral over A^H. Besides, if H is pointed then $A^H = A^{G(H)}$.*

Proof. Using the method of extension of the base field to its algebraic closure we can reduce the situation to pointed Hopf algebras. Then by theorem 1.10 algebra A is fully integral over $A^{G(H)}$ ($\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H \geq |G|$ and thus does not divide $|G|$). As in [13] consider Hopf ideal I generated by the elements of the form $g - 1$ ($g \in G(H)$). H is commutative, thus for all $g \in G(H)$, $x \in H$, $a \in A^{G(H)}$ $\quad gx(a) = xg(a) = x(a)$, therefore $H(A^{G(H)}) \subseteq A^{G(H)}$. It is obvious that $IA^{G(H)} = 0$, so action of H/I on $A^{G(H)}$ is correctly defined. By [6] proposition 5.3.5, H/I is pointed, since the canonical projection $\pi : H \to H/I$ is surjective. Moreover, $G(H/I) = \pi(G(H)) = \{1\}$, thus H/I is *connected* Hopf algebra (the coradical is one-dimensional). Consider action of connected Hopf algebra $Q = H/I$ on $A^{G(H)}$. We will show by induction on m that if $x \in Q^+_{m+1}$ then $x(A^{G(H)}) = 0$. By (2)

$$\Delta(x) = x \otimes 1 + 1 \otimes x + w, \quad w \in Q^+_m \otimes Q^+_m. \tag{7}$$

As in [1, 13] consider ideal QQ^+_m Q which acts as zero on $A^{G(H)}$ by induction. QQ^+_m is also a coideal. Assume that $x(A^{G(H)}) \neq 0$, therefore $x \notin QQ^+_m$. Let elements $1, x, \dots, x^{\ell-1}$ be linearly independent modulo QQ^+_m and

$$x^\ell = \sum_{j=0}^{\ell-1} \alpha_j x^j + w, \quad w \in QQ^+_m, \tag{8}$$

where undoubtedly $\ell \leq d = \dim Q$. Choose k-basis $e_1, \dots, e_d$ in Q so that elements $e_1, \dots, e_{t-1}$ form k-basis for QQ^+_m, $e_t = 1$ and $e_{t+1} = x, \dots, e_{t+\ell-1} = x^{\ell-1}, d \geq t + \ell - 1$. Apply Δ to equation (8). By (7)

$$(x \otimes 1 + 1 \otimes x)^\ell = \sum_{j=0}^{\ell-1} \alpha_j (x \otimes 1 + 1 \otimes x)^j + w'. \tag{9}$$

Since Q is commutative so $w' \in Q \otimes QQ^+_m + QQ^+_m \otimes Q$. For each integer $q \geq 1$

$$(x \otimes 1 + 1 \otimes x)^q = \sum_{i=0}^{q} \binom{q}{i} x^i \otimes x^{q-i}. \tag{10}$$

Subtracting from (9) equation

$$1 \otimes x^\ell = \sum_{j=0}^{\ell-1} \alpha_j 1 \otimes x^j + 1 \otimes w$$

and using (10) we get that

$$\sum_{i=1}^{\ell} \binom{\ell}{i} x^i \otimes x^{\ell-i} = \sum_{j=1}^{\ell-1} \alpha_j \left[\sum_{i=1}^{j} \binom{j}{i} x^i \otimes x^{j-i} \right] + w'', \tag{11}$$

$$w'' \in Q \otimes QQ^+_m + QQ^+_m \otimes Q.$$

As $\operatorname{char} k = 0$ or $> \dim H \geq \dim Q \geq \ell$, so $\binom{\ell}{1} = \ell \neq 0$. Then it follows from (11) that element $x \otimes x^{\ell-1} = e_{t+1} \otimes e_{t+\ell-1}$ in $Q \otimes Q$ is a linear combination of elements $e_s \otimes e_{s'}$, where

either $s < t+1$ or $s' < t+\ell-1$. But it is impossible, since elements $e_q \otimes e_{q'}$, $q, q' = 1, \dots, d$, form k-basis in $Q \otimes Q$. Therefore $x(A^{G(H)}) = 0$, $A^H = \left(A^{G(H)}\right)^Q = A^{G(H)}$ and A is fully-integral over A^H. $\square$

The fact that $A^H = A^{G(H)}$ for pointed commutative Hopf algebras with $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H$ leads us to the idea that these Hopf algebras are pointed. Really, more general fact is true. Prove auxiliary

Proposition 3.2 *If H is connected commutative Hopf algebra and* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > \dim H$ *then $H = k$.*

Proof. By theorem 3.1 for any action of H on any algebra A we have that $A^H = A^{G(H)} = A$, i.e. H always acts trivially. If V is a left H-module then $T(V)$ (tensor algebra on V) has the natural structure of H-module algebra via

$$h(v_1 \otimes \ldots \otimes v_m) = \sum_h (h_{(1)}(v_1)) \otimes \ldots \otimes (h_{(m)}(v_m)).$$

H is a left module over itself, thus $T(H)$ is H-module algebra. $H = k \oplus \ker \epsilon$ as linear spaces; if $x \in \ker \epsilon$ then for each $h \in H$ it is true that $x(h) = \epsilon(x)h = 0$, in particular,

$$x(1_H) = x \cdot 1_H = 0, \tag{12}$$

i.e. $x = 0$ and $H = k$. Here one should not miss the formal unity $1_{T(H)}$ in $T(H)$ with 1_H in H ("$\cdot$" in (12) denotes multiplication in H). $\square$

Prove the main result of this section.

Theorem 3.3 *Let H be a commutative Hopf algebra and* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > \dim H$*. If the coradical H_0 of H is a sub-Hopf algebra then H is cosemisimple, i.e. $H = H_0$; in particular if H is pointed then $H = kG$ – group Hopf algebra.*

Proof. Consider ideal $I = HH_0^+$ in H, it is also a Hopf ideal in H:

$$\Delta(HH_0^+) \subseteq \Delta(H)\Delta(H_0^+) \subseteq (H \otimes H)(H_0^+ \otimes H + H \otimes H_0^+) = HH_0^+ \otimes H + H \otimes HH_0^+,$$

$$S(HH_0^+) = S(H_0^+)S(H) = H_0^+ H = HH_0^+,$$

because H is a commutative Hopf algebra.

By proposition 5.3.5 [6] $(H/I)_0 \subseteq \pi(H_0)$, where $\pi : H \to H/I$ is canonical projection of Hopf algebras, thus H/I is connected commutative Hopf algebra. Note that $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H \geq \dim H/I$. By proposition 3.2 $H/I = k$, therefore $I = HH_0^+ = \ker \epsilon$. Show that it is possible only if $H = H_0$. For this purpose we use well-known result of Nichols and Zoeller that finite-dimensional Hopf algebra is a free left (or right) module over any sub-Hopf algebra [7], more precisely, the stronger result:

Theorem 3.4 ([10]; [6] Theorem 3.3.1) *Let $K \subset H$ be finite-dimensional Hopf algebras. Then*

1. *$H \cong K \otimes H/(K^+H)$ as left K-module and right $H/(K^+H)$-comodule,*

2. $H \cong H/(HK^+) \otimes K$ *as right* K*- and left* $H/(HK^+)$*-comodule.*

In case $K = H_0$ from this theorem it follows that

$$H \cong H/(HH_0^+) \otimes H_0 = H/I \otimes H_0 = k \otimes H_0 = H_0,$$

what was to be shown. $\square$

Next example shows that we can not drop the restrictions on $\operatorname{char} k$ in theorem 3.3.

Example. Let G be any commutative group. We construct Hopf algebra as an extension of kG. Call it $H_{G,N}$, where G is a given group and $N \geq 2$ – integer number. Let $\operatorname{char} k = p \mid N$. As algebra (commutative) $H_{G,N}$ is generated by the group G and element x. We put $x^N = 0$ and for any $g \in G \quad gx = xg$. Define

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x,$$

and kG (contained in $H_{G,N}$) is a group Hopf algebra.

$$\Delta(x^N) = (\Delta(x))^N = (1 \otimes x + x \otimes 1)^N = \sum_{k=0}^{N} \binom{N}{k} x^k \otimes x^{N-k} = 0,$$

because $p \mid N \mid \binom{N}{k}$ when $k \neq 0, N$, and in the rest two summands $x^N = 0$. So Hopf algebra structure on $H_{G,N}$ is correctly defined. $H_{G,N}$ is pointed, $(H_{G,N})_0 = kG$, the basis of $H_{G,N}$ is elements of the form $x^n g$, where $g \in G, \quad n = 0, \ldots, N-1$, $\dim H_{G,N} = |G|N \geq \operatorname{char} k > 0$, but $H_{G,N}$ is not cosemisimple.

In conclusion we get one more structure result about commutative Hopf algebras. It is a direct consequence of theorem 3.3.

Theorem 3.5 *Let* H *be commutative Hopf algebra and* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > \dim H$*. If coradical* H_0 *of* H *is cocommutative then* H *is cosemisimple, and thus cocommutative. In particular, if the base field* k *is algebraically closed then* $H \cong kG$ *for suitable commutative group* G*.*

Proof. Let K be algebraic closure of the base field k. Consider $\overline{H} = H \otimes K$ – Hopf algebra over K. Since K is algebraically closed then $\overline{H}$ is a pointed Hopf algebra, $\dim_K \overline{H} = \dim_k H$, and by theorem 3.3, $\overline{H}$ is a group Hopf algebra and $\overline{H} = \left(\overline{H}\right)_0$. But since $H \otimes K = \overline{H} = \left(\overline{H}\right)_0 \subseteq H_0 \otimes K$, then $H = H_0$ and H is cosemisimple. The second assertion of the theorem is obvious. $\square$

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