Green-Wins Solitaire Revisited — Simultaneous Flips that Affect Many Edges*

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Abstract

We study the Green-Wins Solitaire game, which is a single player edge flipping game played on a given edge-colored geometric triangulation. An edge is flippable if it is a diagonal of a convex quadrilateral, and a flip replaces it by the other diagonal of that quadrilateral. Initially all edges are colored black. A move consists of flipping a black edge and coloring the resulting new edge along with all four edges of the surrounding convex quadrilateral green. The goal is to maximize the number of green edges. We show that in every triangulation on n vertices, for n sufficiently large, at least a fraction of $5/18 \approx 0.277$ edges can be colored green. On the other hand, there exist infinitely many triangulations in which no more than a $1/3$ fraction of edges can be colored green. These results improve earlier bounds of Aichholzer et al. [1].

1 Introduction

In this paper, the term *triangulation* denotes a maximal geometric plane graph: all edges are realized as straight-line segments and all bounded faces are triangles. Conversely, a *triangle* in a triangulation is a bounded facial triangle. Aichholzer et al. [1] studied various games related to triangulations, in particular, the *Green-Wins Solitaire* game. This game is played on a given triangulation, which we consider as edge-colored.

Edge flips. An edge in a triangulation is *flippable* if the union of the two incident faces forms a convex quadrilateral. *Flipping* a flippable edge amounts to replacing said edge by the other diagonal of the surrounding convex quadrilateral; see Figure 1. We say that these five edges, the flipped edge and the four edges of the surrounding quadrilateral, are *affected* by the flip. Edge flips are among the most prominent and well-studied operations for local modification of triangulations and, more generally, planar subdivisions. They serve as a crucial tool in many applications, such as counting and sampling, or optimization, for instance, to compute Delaunay triangulations; see, e.g., the survey by Bose and Hurtado [2].

* This work was started at the 13th Gremo's Workshop on Open Problems (GWOP), June 1–5, 2015, in Feldis (GR), Switzerland. We thank May Szedlák and Antonis Thomas for helpful discussions.
† Research partly supported by the Swiss National Science Foundation within the collaborative DACH project *Arrangements and Drawings* as SNSF Project 200021E-171681.
‡ Research partly supported by Ministry of Education, Science and Technological Development, Republic of Serbia, and Provincial Secretariat for Higher Education and Scientific Research, Province of Vojvodina.

35th European Workshop on Computational Geometry, Utrecht, The Netherlands, March 18–20, 2019.

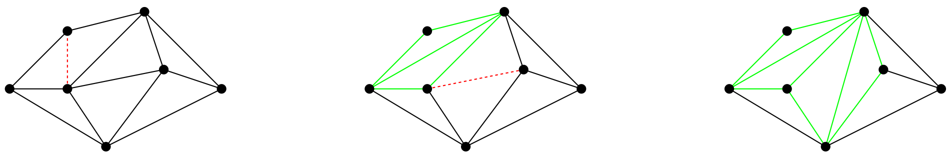


Figure 1 A sequence of moves in Green-Wins Solitaire. The flipped edge is shown red dotted.

Green-Wins Solitaire. Initially all edges are colored black. A move consists of picking a flippable black edge. This edge is flipped, and then all edges that are affected by the flip are colored green; see Figure 1. As green edges cannot be picked anymore, the set of edges flipped over the course of the game is *simultaneously flippable* [3], that is, the convex quadrilaterals surrounding the flipped edges are pairwise interior-disjoint. It also follows that the order of edge flips in a game is irrelevant, and we can describe every strategy as a set of (simultaneously flippable) edges.

For a triangulation T let $\gamma(T)$ denote the ratio of edges of T that can be colored green. Similarly, let $\gamma := \inf_T \gamma(T)$, where T is sufficiently large so as to exclude trivial cases like a single triangle or a K_4 (where no edge can be flipped). Aichholzer et al. [1] show that $1/6 \leq \gamma \leq 5/9$ and specifically ask whether $\gamma \geq 1/2$. They also show that an optimal set of edges to flip can be computed in linear time for convex point sets.

Improvements. The lower bound $\gamma \geq 1/6$ uses a lower bound for the number of simultaneously flippable edges in any triangulation by Galthier et al. [3]. Later Souvaine et al. [7] improved this bound by showing that in any (geometric) triangulation on n vertices at least $(n-4)/5$ edges are simultaneously flippable, which is best possible in general [3]. Plugging this bound into the argument of Aichholzer et al. [1] immediately gives $\gamma \geq 1/5 = 0.2$. Our goal in the following is to further improve this lower bound to $5/18 \approx 0.277 > 1/4$.

▶ **Theorem 1.** *In every triangulation of n points, for n sufficiently large, there exists a simultaneously flippable set of edges that affects at least a $5/18$ fraction of all edges.*

Before attacking the lower bound, let us note that the upper bound $\gamma \leq 5/9$ can be easily improved by considering families of triangulations for which the lower bound on the number of simultaneously flippable edges is tight (see Figure 2). It must have been an oversight that this was not observed by Aichholzer et al. [1] because the upper bound on the number of simultaneously flippable edges [3] was already known at that time.

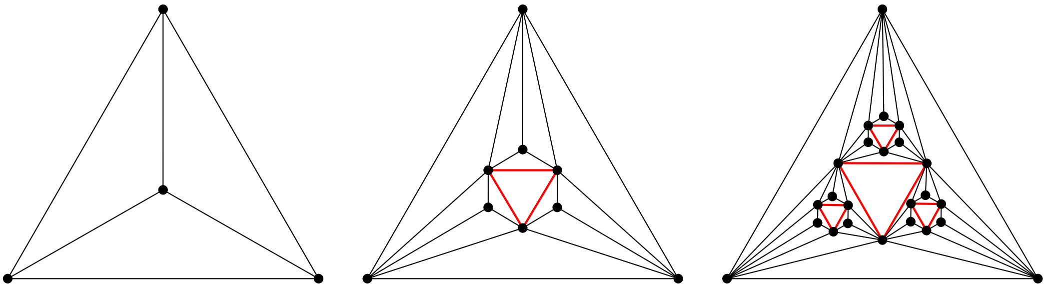
▶ **Observation 2.** *For infinitely many $n \in \mathbb{N}$, there exists a triangulation on n points such that every simultaneously flippable set of edges affects at most $n-4$ out of $3n-6$ edges.*

In summary we have $5/18 \leq \gamma \leq 1/3$. The rest of the paper is devoted to derive the lower bound and prove Theorem 1.

2 Preliminaries

Sets of triangles. Consider a triangulation T on $n \geq 5$ vertices and a set U of triangles in T. The unbounded face is bounded by a cycle of r vertices that form the convex hull, where $3 \leq r \leq n$. By the Euler Formula, T has $3n-r-3$ edges and $2n-r-1$ faces, all but one of which are triangles. A vertex or edge of T is *interior* if it is not on the convex hull.

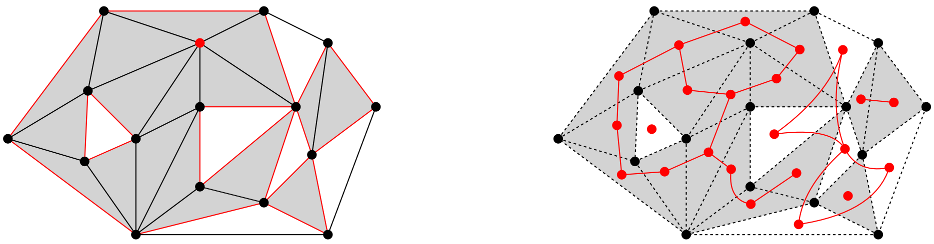
A vertex or edge of T is (1) *internal* to U if all incident faces are in U, (2) *external* to U if no incident face is in U, or (3) on the *boundary* of U if it is incident to at least one face



(a) (b) (c)

Figure 2 The upper bound construction for the number of simultaneously flippable edges by Souvaine et al. [7]. Recursively the central vertex of K_4 subconfigurations is replaced by a hexagon, connected as shown in (b). All flippable edges are shown in red; they appear in triangles. In each such triangle, no more than one edge can be selected for any simultaneous flip.

in U and at least one face that is not in U. Note that a vertex or edge on the convex hull is not internal to U by definition. See Figure 3a for illustration.



(a) U **(b)** $T^*|_U$

Figure 3 (a) A set U of gray triangles with 1 internal vertex (red), 16 internal edges, 3 components, 1 hole, and 19 boundary edges (red). Confirming Alpaca 3 we have $3 \cdot 1 + 3 \cdot 1 + 19 = 16 + 3 \cdot 3$.

Let $T^*|_U$ denote the following graph on the triangles of T: Two triangles of T are connected in $T^*|_U$ if (1) they share an edge and are both in U or (2) they share a vertex and are both not in U. A *component* of U is a component of U in $T^*|_U$. A *hole* in U is a component of $T^*|_U \setminus U$ that is contained inside a cycle of triangles from U in $T^*|_U$. We obtain the following variation of the Euler Formula for sets of triangles in a triangulation.[1]

▶ **Alpaca 3.** *Consider a triangulation T and a subset U of triangles of T. Then*

$$3h + 3v_i + e_b = e_i + 3c,$$

where h denotes the number of holes in U, v_i denotes the number of internal vertices of U, e_b denotes the number of boundary edges of U, e_i denotes the number of internal edges of U and c denotes the number of components of U in $T^*|_U$.

Strategy. Let F denote a maximum size set of simultaneously flippable edges in T, and let U denote the set of triangles in T that are incident to an edge in F. We flip the $f := |F|$

[1] It is named after the cute animals that the authors watched when working on this problem and they realized: It is not a lama but an alpaca...

edges in F and color them green. For every flip, the four edges of the bounding quadrilateral are also colored green. However, each of these edges may be colored green twice overall in case the edge is incident to two triangles from U. Exactly the e_b edges on the boundary of U are colored once only. Therefore, the number of green edges after flipping F is

$$f + 4f/2 + e_b/2 = 3f + e_b/2. \tag{GF}$$

In order to bound this quantity, we want to obtain a good lower bound for e_b to combine with the known lower bound $f \geq (n-4)/5$.

▶ **Lemma 4.** *We have $e_b \geq 4c$.*

Proof. As T is a triangulation, every component of U in $T^*|_U$ is bounded by at least three edges. Furthermore, every component of U consists of pairs of triangles and hence an even number of triangles. A triangulation whose outer boundary forms a triangle corresponds to a maximal planar graph, which has $2n-5$ bounded faces—an odd number. Therefore, no component of U has a triangle as an outer boundary. ◀

3 Counting black edges: Proof of Theorem 1

We color the edges of T as follows. An edge e of T is colored *green* if it is incident to a triangle from U; otherwise, e is colored *black*. Note that we work with the original triangulation T, no edges are flipped. If a black edge was flippable, then we could flip it to color more edges green, contradicting the maximality of F. Hence no black edge is flippable. As a first step we observe that not too many vertices, relatively speaking, are on the convex hull.

▶ **Lemma 5.** *For every $\alpha \in (0,1)$ and $r \geq \frac{3\alpha}{1+\alpha}n$, at least an $(\alpha - \varepsilon)$ fraction of the edges in T are green, where $\varepsilon = \varepsilon(n)$ tends to zero when $n \to \infty$.*

Proof. At least $r-3$ edges are flippable in T [4]. Clearly, every flippable edge can be colored green: if it is not, then flip it. Hence, at least a fraction of $(r-3)/(3n-r-3)$ edges is colored green. This expression is monotonically increasing in r, for $3 \leq r \leq n$. Therefore,

$$\frac{r-3}{3n-r-3} \geq \frac{\frac{3\alpha}{1+\alpha}n - 3}{3n - \frac{3\alpha}{1+\alpha}n - 3} = \frac{\alpha n - (1+\alpha)}{n - (1+\alpha)} \geq \alpha - \frac{2}{n-2}$$

which converges to α, for $n \to \infty$. ◀

In the following, we investigate the subgraph B of T that is induced by the black edges. The vertices of B fall into three groups: vertices on the convex hull, vertices on the boundary of U, and *internal* vertices (vertices that are not on the convex hull and not incident to any green edge). An edge of B is *internal* if it is incident to two triangles in B. In particular, an internal edge is not a convex hull edge.

Following standard terminology [6], we call an edge $e = uv$ of T *separable* at its endpoint u if there exists a line ℓ through u so that e is the only edge of T incident to u on one side of ℓ. In other words, u is *pointed* (has a free angle $\geq \pi$) in $T \setminus e$. We observe (cf. [5, 6]):

(S1) Every unflippable edge is separable at one of its endpoints and only the convex hull edges are separable at both endpoints.

(S2) At an interior vertex v of degree ≥ 4, at most two incident edges are separable; if two incident edges are separable, then they are consecutive in the circular order around v.

In particular, (S1) implies that an edge that is incident to two convex hull vertices is either a convex hull edge or flippable. So no internal edge of B connects two convex hull vertices. By the following lemma, all internal vertices of B have degree 3 (in B and T).

▶ **Lemma 6** ([5, Lemma 4.2]). *In a triangulation, every interior (not on the convex hull) vertex of degree four or higher is incident to a flippable edge.*

In a triangulation, no two interior degree three vertices are adjacent. Let us bound the number of black edges. Every black edge connects two vertices from the abovementioned three groups: At most r edges are convex hull edges, and every edge that is incident to an internal vertex is incident to exactly one internal vertex (because every internal vertex has degree three). All remaining edges are incident to at least one vertex on the boundary of U but not on the convex hull (because there is no internal black edge between any two convex hull vertices). Denote the number of these remaining edges by $\overline{e_i}$, denote by d_3 the number of internal (hence degree-3) vertices of B, and denote by e_C the number of convex hull edges on the boundary of U. We select F so that the number d_3 is smallest over all maximum size sets of simultaneously flippable edges in T.

▶ **Lemma 7.** *We have $3d_3 + e_C \leq 2e_b$.*

Let us derive a bound for $\overline{e_i}$ by applying Alpaca 3 to $T \setminus U$. The boundary of $T \setminus U$ is formed by edges that are on the convex hull or on the boundary of U, and there are $(e_b - e_C) + (r - e_C) = e_b + r - 2e_C$ such edges. The edges incident to the d_3 internal vertices of B are not counted in $\overline{e_i}$, and so we have no internal vertices to consider. Every hole in $T \setminus U$ corresponds to at least one separate component of U, which in turn corresponds to a separate component of U in T, and so the number of holes is upper bounded by c. Altogether we obtain

$$3c + e_b + r - 2e_C \geq \overline{e_i}. \tag{1}$$

▶ **Lemma 8.** *There are no more than $3e_b + 3c + 2r - 4e_C$ black edges.*

Proof. We have exactly $r - e_C$ black convex hull edges, exactly $3d_3$ edges that are incident to an interior vertex, and exactly $\overline{e_i}$ other edges. Using Lemma 7 and (1), we can bound the number of black edges as $(r - e_C) + 3d_3 + \overline{e_i} \leq (r - e_C) + (2e_b - e_C) + (3c + e_b + r - 2e_C)$. ◀

▶ **Lemma 9.** *There are at least $3(n - r - e_b - c - 1) + 4e_C$ green edges.*

Proof. The total number of edges in T is $3n - r - 3$. Thus, by Lemma 8, we have at least $(3n - r - 3) - (3e_b + 3c + 2r - 4e_C) = 3(n - r - e_b - c - 1) + 4e_C$ green edges. ◀

If we head for a bound of $\gamma \geq \alpha$, then we may assume that

$$c \leq 3n\left(\frac{\alpha}{2} - \frac{1}{10}\right) - \frac{\alpha}{2}r.$$

Otherwise, by Lemma 4 and the lower bound $f \geq (n-4)/5$, we have

$$3f + \frac{1}{2}e_b \geq \frac{3}{5}n + 2c \geq \frac{3}{5}n + 3n\left(\alpha - \frac{1}{5}\right) - \alpha r \geq \alpha(3n - r - 3),$$

and we are done. In combination with (GF) and Lemma 9 we get

$$\begin{aligned} 3f + \frac{1}{2}e_b &\geq 3(n - r - e_b - c) \\ \frac{7}{2}e_b &\geq 3(n - r - f - c) \\ \frac{1}{2}e_b &\geq \frac{3}{7}(n - r - f - c) \end{aligned}$$

and, therefore,

$$\begin{aligned}
3f + \frac{1}{2}e_b &\geq 3f + \frac{3}{7}(n - r - f - c)\\
&= \frac{18}{7}f + \frac{3}{7}(n - r - c)\\
&\geq \frac{18}{35}n + \frac{3}{7}(n - r - c)\\
&= \frac{33}{35}n - \frac{3}{7}(r + c)\\
&\geq \frac{33}{35}n - \frac{3}{7}r - \frac{3}{7}\left(3n\left(\frac{\alpha}{2} - \frac{1}{10}\right) - \frac{\alpha}{2}r\right)\\
&= \frac{1}{70}\Big((75 - 45\alpha)n - (30 - 15\alpha)r\Big).
\end{aligned}$$

Thus, the fraction of green edges is at least

$$\frac{3f + \frac{1}{2}e_b}{3n - r} \geq \frac{(75 - 45\alpha)n - (30 - 15\alpha)r}{70(3n - r)} = \frac{1}{70}\left(30 - 15\alpha - \frac{15n}{3n - r}\right).$$

The fraction $\frac{-15n}{3n-r}$ is monotonically decreasing as a function of r and so it is minimized for r maximal, that is, $r = \frac{3\alpha}{1+\alpha}n$, due to Lemma 5. In this case, we obtain a fraction of

$$\frac{1}{70}\left(30 - 15\alpha - \frac{15n}{3n - \frac{3\alpha}{1+\alpha}n}\right) = \frac{1}{70}\Big(30 - 15\alpha - 5(1 + \alpha)\Big) = \frac{1}{14}(5 - 4\alpha).$$

Optimizing for α yields $\gamma \geq \alpha = 5/18 > 0.277$, which completes the proof of Theorem 1.

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