On Regular Vertices of the Union of Planar Convex Objects*

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ABSTRACT

Let $\mathcal{C}$ be a collection of n compact convex sets in the plane, such that the boundaries of any pair of sets in $\mathcal{C}$ intersect in at most s points, for some constant s. We show that the maximum number of *regular* vertices (intersection points of two boundaries that intersect twice) on the boundary of the union U of $\mathcal{C}$ is[1] $O^*(n^{4/3})$, which improves earlier bounds due to Aronov *et al.* [4]. The bound is nearly tight in the worst case.

Categories and Subject Descriptors

F.2.2 [**Theory of Computation**]: Analysis of Algorithm and Problem Complexity Nonnumerical Algorithms and Problems [geometrical problems and computations]

General Terms

Theory

Keywords

Union of geometric objects, Regular vertices, $(1/r)$-cuttings

1. INTRODUCTION

Let $\mathcal{C}$ be a collection of n compact convex sets in the plane as in the abstract. Let U denote the union of $\mathcal{C}$. If the boundaries of a pair of sets in $\mathcal{C}$ intersect exactly twice, we refer to their two intersection points as *regular* intersections; all other boundary intersections are called *irregular*. Several recent papers have considered the problem of obtaining sharp bounds on the number of regular intersection points that can appear on the boundary of the union U. In the simplest instance of this problem, we assume that the boundaries of any pair of sets in $\mathcal{C}$ intersect at most twice, but make no other assumption on the shape of these sets; we then refer to $\mathcal{C}$ as a collection of *pseudo-disks*. In an early paper [10], Kedem *et al.* show that in this case the boundary of the union contains at most $6n-12$ intersection points, and this bound is tight in the worst case. Pach and Sharir [13] have shown that, for the special case where $\mathcal{C}$ consists of convex sets, one always has $R \le 2I + 6n - 12$, where R (resp., I) denotes the number of regular (resp., irregular) points on ∂U, thus generalizing the result of Kedem *et al.*, in which $I = 0$.

The bound of Pach and Sharir is tight in the worst case, but since I can be large, it does not provide a good "absolute" upper bound (a bound that depends only on n) on R. In fact, I can be $\Omega(n^2)$ in the worst case, and there exist constructions in which both I and R are $\Theta(n^2)$ (see Figure 1(a)). However, in these lower bound constructions, some pairs of the boundaries of the sets in $\mathcal{C}$ intersect in an arbitrarily large number of points (that is, the assumption in the abstract does not hold). It is therefore interesting to seek bounds on R that are independent of I and depend only on n, in cases where each pair of boundaries intersect in a constant number of points. This has been done by Aronov *et al.* [4]. Under similar assumptions as in the abstract, they obtained the upper bound $R = O^*(n^{3/2})$. For the more general case, where the sets in C are not necessarily convex, they show the existence of a positive constant δ, which depends only on s, so that $R = O(n^{2-\delta})$.

Our result. In this paper we consider the case where $\mathcal{C}$ satisfies the assumptions in the abstract, and derive an improved bound on R. Specifically, we show that $R = O^*(n^{4/3})$. This improves the first bound of [4]. Moreover, this bound is nearly tight in the worst case, since one can easily construct n rectangles and disks which generate $\Theta(n^{4/3})$ regular vertices on the boundary of their union; see [13] for details.

Besides being an intriguing problem in itself, which is finally fully resolved (for the convex case), it arises in appli-

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[1] In this paper, a bound of the form $O^*(f(n))$ means that the actual bound is $C_\varepsilon f(n) \cdot n^\varepsilon$, for any $\varepsilon > 0$, where C_ε is a constant that depends on ε (and generally tends to ∞ as ε decreases to 0).

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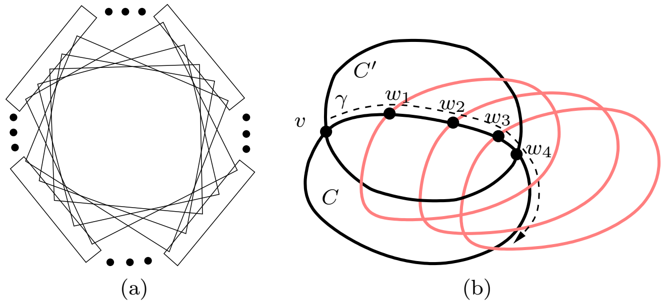

Figure 1: (a) A construction with $R = \Theta(n^2)$. (b) A charging scheme for bounding the complexity of the union. The vertex v, generated by the intersection boundaries of C and C', is charged to the block of the four vertices $w_1, \ldots, w_4$, appearing along the boundary portion γ of C encountered when tracing ∂C from v as depicted.

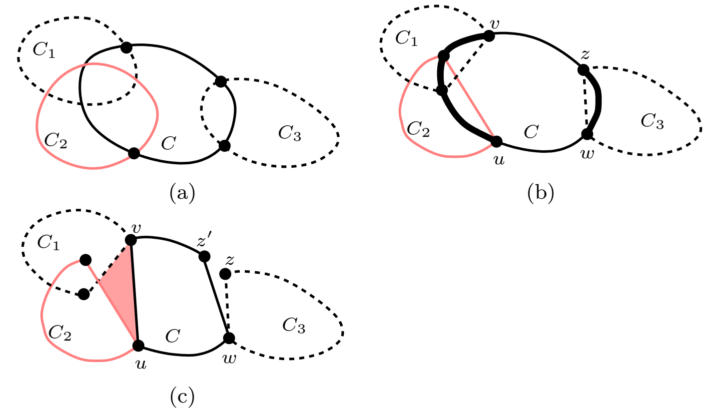

Figure 2: Demonstration of the transformation rule. (a) ∂C creates with each of the three sets C_1, C_2, C_3 regular vertices on the boundary of the union. (b) Each of C_1, C_2, C_3 is shrunk by the chords connecting its intersections with ∂C. (c) Shrinking C by similar shortcuts; wz is replaced by a nearby chord wz' to make C and C_3 touch at a single point. The shaded region is a new connected component of the complement of the union (a new "hole").

cations that seek bounds on the complexity of the union of geometric objects in two and three dimensions, such as robot motion planning [8] and solid modeling. One technique for analyzing this complexity is via a *charging scheme*, where we start from a vertex v of the union, and follow one of the two incident boundaries, into the interior of the union, hoping to collect at least k vertices of the arrangement of the boundaries before reaching the union boundary again, where k is some prescribed parameter; see Figure 1(b) and [15, 16]. Such a charging scheme will fail if we return to the union boundary before collecting enough vertices; the worst case of which is when we return immediately to the union boundary. In many cases this situation can be handled by "blaming" it to a regular vertex on the union boundary, and having a sharp bound on the number of such vertices is then crucial for the success of the scheme.

2. ANALYSIS

Let $\mathcal{C}$ be a collection of n compact convex sets in the plane, each pair of whose boundaries intersect in at most s points, for some constant s. For each $C \in \mathcal{C}$, the segment connecting the leftmost and rightmost points of C is called the *spine* of C; we assume (without loss of generality) that it is unique, and denote it by σ_C (note that σ_C is contained in C, due to its convexity).

As already defined, a pair C, C' of sets in $\mathcal{C}$ are said to intersect *regularly* if $|\partial C \cap \partial C'| = 2$. Each of these two intersection points is called a *regular* vertex of the arrangement $\mathcal{A}(\mathcal{C})$ of (the boundary curves of the sets in) $\mathcal{C}$.

We establish an upper bound on the maximum number of regular vertices on the *boundary* of the union U of $\mathcal{C}$, which improves the earlier bound $O^*(n^{3/2})$, due to Aronov *et al.* [4]. Specifically, we show

THEOREM 2.1. *Let $\mathcal{C}$ be a set of n compact convex sets as above. Then the number of regular vertices on the boundary of the union of $\mathcal{C}$ is at most $O^*\left(n^{4/3}\right)$, when the small hidden factor in this bound depends on s. This bound is nearly worst-case tight, that is, there are constructions that yield $\Omega\left(n^{4/3}\right)$ regular vertices that appear on the boundary of the union (already for $s = 4$).*

Overview of the proof. We first assume that the given sets in $\mathcal{C}$ are in general position. The proof proceeds through the following stages. We first apply the transformation of [4] to the collection $\mathcal{C}$ of convex sets. The transformed sets satisfy the following properties (see [4, Lemma 1] and Figure 2 for further details): (i) They are convex. (ii) Any two boundaries intersect at most s times. (iii) Any two sets $C, C' \in \mathcal{C}$ that intersected regularly (before the transformation) either become disjoint or *touch* at a single point. More precisely, if C, C' intersected regularly with at least one point of intersection of their boundaries on ∂U, the transformed sets are openly disjoint and touch each other at a point on ∂U. If they intersected regularly without creating vertices on ∂U, they are now disjoint. To simplify the notation, we let $\mathcal{C}$ denote from now the set of the transformed regions.

Note that the spines of the transformed sets may be different from those of the original sets. Note also that, after this transformation, any regular vertex on the boundary of the union must be formed by a pair of sets whose spines are disjoint.

We then apply a decomposition scheme that consists of two phases. The first phase represents all pairs of sets of $\mathcal{C}$ with disjoint spines, so that one of these spines lies below the other (see below for a precise definition), as the disjoint union of complete bipartite graphs, whose overall complexity is sufficiently small, in a sense to be made precise below.

We then fix one such complete bipartite subgraph $A \times B$, where the spines of the sets in A all lie below those of the sets of B, and analyze the number of regular vertices that it contributes to the union boundary. A crucial property of such a graph is that each of these regular vertices must lie either on the upper envelope of the top boundaries of sets whose spines are in A, or on the lower envelope of the bottom boundaries of sets whose spines are in B. We then form, say, the upper envelope E_A^+ of the top boundaries of the sets in A, and decompose it into maximal connected arcs, each contained in the boundary of a single set, and having disjoint x-spans.

The fact that regular vertices are formed by touching pairs, suggests a second decomposition phase, in which we transform $A \times B$ into a union of complete bipartite subgraphs, such that each such subgraph $A' \times B'$ is associated with some vertical strip Σ, and each spine σ of a set in B' lies, within the strip Σ, above every arc δ whose incident set belongs to A'. It is then easy to show that the number of regular vertices of the union, induced by pairs of sets in $A' \times B'$, is only nearly-linear in $|A'| + |B'|$.

The second decomposition phase is somewhat involved. It consists of decomposition steps that alternate between the primal and dual planes, where each step is based on a cutting of a certain line arrangement. While the dual decomposition is more "conventional", the primal one is trickier, and requires careful analysis of the way in which the arcs δ interact with the spines from the other set.

Finally, using these bounds, we put everything together and obtain the bound asserted in the theorem.

Transforming the sets. We begin by applying to $\mathcal{C}$ the transformation of Aronov *et al.* [4], as explained in the overview. We continue to denote by $\mathcal{C}$ the collection of the shrunk sets.

Let C and C' be two members of $\mathcal{C}$ that touch each other at a point that lies on ∂U. Clearly, as already noted, σ_C and $\sigma_{C'}$ are disjoint, and one of them, say, σ_C, *lies below* the other, which means that (i) their x-spans have nonempty intersection J; (ii) σ_C lies below $\sigma_{C'}$ at each $x \in J$.

The first bi-clique decomposition. We collect all pairs of spines so that one of them lies below the other, as the disjoint union of complete bipartite graphs (*bi-cliques*), so that the overall size of their vertex sets is $O^*(n^{4/3})$. More precisely, the following stronger property holds.

LEMMA 2.2. *Given a collection $\mathcal{C}$ as above, let G be the graph whose vertices are the regions in $\mathcal{C}$, and whose edges connect pairs of regions (C, C'), such that σ_C lies below $\sigma_{C'}$. Then there exists a decomposition $G = \bigcup_i A_i \times B_i$ into pairwise edge-disjoint bi-cliques, such that*

$$\sum_i \left(|A_i|^{2/3}|B_i|^{2/3} + |A_i| + |B_i|\right) = O^*(n^{4/3}). \qquad (1)$$

Proof: This is a standard result in "batched" range searching, and can be found, e.g., in [1, 2]; see also [12]. The proof is given below for the sake of completeness.

Let $\sigma = pq$, $\sigma' = p'q'$ be a pair of spines such that σ lies below σ'. This relationship can be expressed as the disjunction of a constant number of conjunctions of above/below relationships, over the possible x-orders of p, q, p' and q', where each atomic relationship asserts that an endpoint of one spine lies above or below the line containing the other spine. For example, if the x-order of the endpoints is p, p', q, q', then we require that p' lie above the line ℓ containing σ and that q lie below the line ℓ' containing σ'; see Figure 3(a)–(c). For simplicity of exposition, we describe the construction only for the subgraph of G that consists of pairs of regions with this property; all other subcases are handled in a fully symmetric manner.

We apply a multi-level decomposition scheme, where each level produces a decomposition into bi-cliques that satisfy some of the constraints, and each of them is passed to the next level to enforce additional constraints. At the two top levels, we produce a collection of pairwise edge-disjoint bi-cliques, such that, for each of these graphs $A_1 \times B_1$, for each spine $\sigma = pq \in A_1$ and for each spine $\sigma' = p'q' \in B_1$, the x-order of the endpoints is p, p', q, q', and such that the union of these graphs gives all such pairs of spines. This is easily done using a 2-dimensional range tree construction [1, 7]. The sum of the vertex sets of the resulting subgraphs is $O(n \log^2 n)$. Moreover, (1) is easily seen to hold for the decomposition thus far.

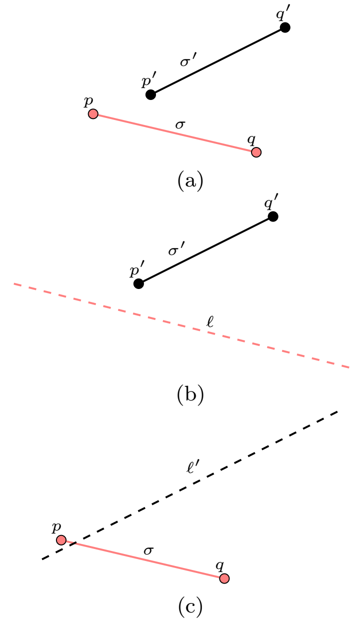

Figure 3: (a) The two spines σ, σ' are disjoint and σ lies below σ'. (b) The left endpoint p' of σ' lies above the line ℓ containing σ, and (c) the right endpoint q of σ lies below the line ℓ' containing σ'.

The next level enforces, for each resulting subgraph $A_1 \times B_1$, the condition that p' lie above the line ℓ containing σ, for $\sigma \in A_1$ and p' the left endpoint of a spine $\sigma' \in B_1$. Put $m_1 = |A_1|$ and $n_1 = |B_1|$. For this we choose a sufficiently large constant parameter r, and construct a $(1/r)$-cutting [11] of the arrangement of the lines that contain the spines of A_1. We obtain $O(r^2)$ cells, each of which is crossed by at most m_1/r lines, and contains at most n_1/r^2 left endpoints of spines of B_1. (The latter property can be enforced by further splitting some cells of the cutting; also, assuming general position, we can construct the cutting so that no endpoint of any spine lies on the boundary of any cutting cell.) For each cell Δ, we form the bi-clique $A_2'(\Delta) \times B_2(\Delta)$, where $B_2(\Delta)$ consists of all spines whose left endpoints are in Δ, and where $A_2'(\Delta)$ consists of all spines whose supporting lines pass completely below Δ. These graphs are passed to the next level of the structure. We then consider, for each cell Δ, the set $A_2(\Delta)$ of spines of A_1 that cross Δ, and the set $B_2(\Delta)$ as defined above. We pass to the dual plane, where the lines of the spines in $A_2(\Delta)$ are mapped to points and the left endpoints of spines in $B_2(\Delta)$ are mapped to lines. We construct a $(1/r)$-cutting of the arrangement of these dual lines, obtaining $O(r^2)$ cells, each of which is crossed by at most $|B_2(\Delta)|/r \le n_1/r^3$ lines and contains at most $|A_2(\Delta)|/r^2 \le m_1/r^3$ points. As above, we construct, for each cell of the cutting, a bi-clique from the dual points in the cell and the lines that pass fully above the cell, and pass all these graphs to the next level. We are left with $O(r^4)$ subproblems, each involving at most m_1/r^3 spines of A_1 and at most n_1/r^3 spines of B_1, which we process recursively. We continue to process each subproblem as above, going back to the primal plane, and keep alternating in this manner, until we reach subproblems in which either $m_1^2 < n_1$, or $n_1^2 < m_1$. In the former (resp., latter) case, we continue the recursive construction *only* in the primal (resp., dual) plane, and stop as soon as one of m_1, n_1 becomes smaller than r, in which case we produce a collection of singleton bi-cliques.

Suppose first that $\sqrt{m_1} \le n_1 \le m_1^2$. We show below that, for any fixed initial subgraph $A_1 \times B_1$, the resulting bi-clique decomposition $\{A_2'(\Delta) \times B_2(\Delta)\}_\Delta$, over all cells Δ of all the cuttings, satisfies

$$\sum_\Delta \left(|A_2'(\Delta)|^{2/3}|B_2(\Delta)|^{2/3} + |A_2'(\Delta)| + |B_2(\Delta)|\right) = \qquad (2)$$
$$O^*\left(|A_1|^{2/3}|B_1|^{2/3} + |A_1| + |B_1|\right) \quad ,$$

and the same holds for the corresponding decompositions in the dual spaces.

Indeed, let us consider only the primal decompositions, since the dual ones are handled in exactly the same manner. Since r is taken to be a constant, the sum in (2), over the graphs produced at the top level of the recursion, is at most $C(r)\left(|A_1|^{2/3}|B_1|^{2/3} + |A_1| + |B_1|\right)$, where $C(r)$ is a constant that depends on r. In the next level, we have at most $C'r^4$ subproblems, for some absolute constant $C' > 0$, each involving at most $|A_1|/r^3$ spines of A_1 and at most $|B_1|/r^3$ spines of B_1. The overall contribution to the sum in (2) by the bi-cliques produced at this level is at most

$$C'r^4 \cdot C(r)\left(\left(\frac{|A_1|}{r^3}\right)^{2/3}\left(\frac{|B_1|}{r^3}\right)^{2/3} + \frac{|A_1|}{r^3} + \frac{|B_1|}{r^3}\right) =$$
$$C'C(r)\left(|A_1|^{2/3}|B_1|^{2/3} + |A_1|r + |B_1|r\right).$$

Continuing in this manner, the contribution to the sum in (2) at the j-th level of the recursion is at most

$$(C')^j r^{4j} \cdot C(r)\left(\frac{|A_1|^{2/3}}{r^{2j}}\frac{|B_1|^{2/3}}{r^{2j}} + \frac{|A_1|}{r^{3j}} + \frac{|B_1|}{r^{3j}}\right) =$$
$$(C')^j C(r)\left(|A_1|^{2/3}|B_1|^{2/3} + |A_1|r^j + |B_1|r^j\right),$$

where, at the last level, $r^j = \min\left\{\frac{|A_1|^{2/3}}{|B_1|^{1/3}}, \frac{|B_1|^{2/3}}{|A_1|^{1/3}}\right\}$. Summing over the logarithmically many levels of the recursion, we obtain the overall bound $O^*(|A_1|^{2/3}|B_1|^{2/3})$.

It remains to consider the cases $|A_1|^2 < |B_1|$, $|B_1|^2 < |A_1|$. It suffices to consider only the first case. Hence, after j levels of recursion (only in the primal plane), the contribition to (2) is at most

$$(C'')^j r^{2j} \cdot C(r)\left(\left(\frac{|A_1|}{r^j}\right)^{2/3}\left(\frac{|B_1|}{r^{2j}}\right)^{2/3} + \frac{|A_1|}{r^j} + \frac{|B_1|}{r^{2j}}\right) =$$
$$(C'')^j C(r)\left(|A_1|^{2/3}|B_1|^{2/3} + |A_1|r^j + |B_1|\right),$$

where C'' is another absolute constant, and where the last j satisfies $r^j = O(|A_1|)$. Substituting this value, summing over all j, and using the inequality $|A_1|^2 \le |B_1|$, we get the overall bound $O^*(|B_1|)$.

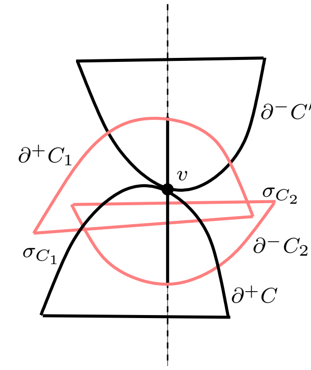

Figure 4: If the vertex v does not lie on either E_A^+ or E_B^-, then it is "hidden" from E_A^+ by $\partial^+ C_1$, and from E_B^- by $\partial^- C_2$, for some $C_1 \in A$, $C_2 \in B$. But then v is contained in (the interior of) $C_1 \cup C_2$, contrary to the construction.

Note that, in the preceding case $\sqrt{|A_1|} \le |B_1| \le |A_1|^2$, when the recursion bottom out, we have sets A', B' that satisfy $|A_1'|^2 \le |B_1'|$ or $|B_1'|^2 \le |A_1'|$, so the same analysis adds to (2) the terms $O^*(|A_1| + |B_1|)$, which thus completes the proof of the claim.

The final level of the structure enforces, for each resulting subgraph $A_2 \times B_2$, the condition that q lie below the line ℓ' containing σ', for $\sigma' \in B_2$ and q the right endpoint of a spine $\sigma \in A_2$. This is done in a fully analogous manner to the preceding step. It is easily checked that, in complete analogy to the preceding analysis, the bi-clique decomposition $\{A_\alpha \times B_\alpha\}_\alpha$, that results from the fixed bi-clique $A_2 \times B_2$, over all cells of all the cuttings, satisfies

$$\sum_\alpha \left(|A_\alpha|^{2/3}|B_\alpha|^{2/3} + |A_\alpha| + |B_\alpha|\right) =$$
$$O^*\left(|A_2|^{2/3}|B_2|^{2/3} + |A_2| + |B_2|\right).$$

Combining this with (2), summing over the entire collection of these last-stage decompositions, and using the fact that (1) holds for the initial-level decomposition, we conclude that (1) holds for the overall final decomposition, thus completing the proof of the lemma. □

Handling a single bi-clique. Fix one of the resulting graphs $A \times B$. All the spines of the sets in A lie below all the spines of the sets in B. Put $n_A = |A|$ and $n_B = |B|$.

Let v be a regular vertex of the union lying on the top boundary $\partial^+ C$ and on the bottom boundary $\partial^- C'$, for two sets $C \in A$, $C' \in B$; clearly, this is the only possible situation. We claim that v lies either on the upper envelope E_A^+ of the top boundaries of the sets in A, or on the lower envelope E_B^- of the bottom boundaries of the sets in B. Indeed, if this were not the case, then v must lie below some top boundary $\partial^+ C_1$, for $C_1 \in A$, and above some bottom boundary $\partial^- C_2$, for $C_2 \in B$; see Figure 4. By construction, σ_{C_1} lies below σ_{C_2} at the x-coordinate x_v of v, which implies that the entire vertical segment connecting $\partial^+ C_1$ and $\partial^- C_2$ at x_v is fully contained in $C_1 \cup C_2$, so v cannot lie on the boundary of the union, a contradiction that establishes the claim.

Without loss of generality, we consider only the case where v lies on the upper envelope E_A^+ of the top boundaries of the sets in A. Since any pair of these boundaries intersect in at most s points, the number $m = m_A$ of connected portions of top boundaries that constitute E_A^+ satisfies $m \le \lambda_{s+2}(n_A)$, where $\lambda_s(q)$ is the maximal length of Davenport-Schinzel sequences of order s on q symbols (see [15]). Enumerate these arcs from left to right as $\delta_1, \ldots, \delta_m$, and let A^* denote the set containing them.

Let H_0 denote the subgraph of $A^* \times B$ consisting of all the pairs (δ, C), such that C forms with (the set of A containing) δ a regular vertex on ∂U (where the two sets touch each other), and so that (a) the touching point lies on δ and on $\partial^- C$, and (b) δ lies fully below σ_C (i.e., the x-span of σ_C contains that of δ). If (b) does not hold, then an endpoint of σ_C lies above δ, and there can be at most two such arcs δ (for any fixed C), so the number of excluded pairs is at most $2n_B$. Hence, the number of regular "bichromatic" vertices formed by $A \cup B$, lying on E_A^+, and not counted in H_0 is only $O(n_B)$.

Our next step is to construct a collection of complete bipartite graphs $\{A_i^* \times B_i\}_i$, such that, for each i, $A_i^* \subset A^*$, $B_i \subset B$, and the union of these graphs is edge-disjoint and covers H_0. In addition: (a) The sum $\sum_i (|A_i^*| + |B_i|)$ will be small, in a sense to be made precise below. (b) For each i, there is an x-interval I_i such that, for each $\delta \in A_i^*$ and $C \in B_i$, the line ℓ_C containing the spine σ_C of C passes fully above δ over I_i (although σ_C may end within I_i). (c) For each pair $(\delta, C) \in H_0$, there exists i such that $\delta \in \mathcal{A}_i^*$, $C \in B_i$, and the x-coordinate of the touching point $\delta \cap \partial^- C$ lies in I_i.

Suppose we have such a collection at hand. Fix one of the graphs $A_i^* \times B_i$. We claim that, for any $\delta \in A_i^*$, $C \in B_i$, such that $(\delta, C) \in H_0$, the relevant touching vertex v of $\delta \cap \partial^- C$ lies on the lower envelope $E_{B_i}^-$ of the bottom boundaries of the sets in B_i. This follows using the same arguments as in the preceding step (see also [4]). That is, suppose to the contrary that v lies above the bottom boundary $\partial^- C'$ of another set $C' \in B_i$. By assumption, $\sigma_{C'}$ lies above v (because $v \in \delta$) and thus v lies in the interior of C', contradicting the assumption that v is a vertex of the union. Note that it is crucial that the x-coordinate of v lies in the x-interval I_i as above; see Figure 5(a).

In other words, each vertex of this kind is an intersection point of $E_{B_i}^-$ and the concatenation of the arcs in A_i^*. Hence, by merging, in the x-order, the breakpoints of $E_{B_i}^-$ and the endpoints of the arcs in A_i^*, it easily follows that the number of such vertices is $O(\lambda_{s+2}(|B_i|) + |A_i^*|) = O^*(|A_i^*| + |B_i|)$. Summing this bound over all subgraphs $A_i^* \times B_i$ yields an overall bound for the number of pairs $(\delta, C) \in H_0$ (excluding the linear number of pairs that we do not count in H_0, as above). See below for the precise bound.

To obtain the desired cover of H_0, we proceed as follows. Let L denote the set of the lines supporting the spines of the sets in B. Fix a sufficiently large constant parameter r, and construct a suboptimal $(1/r)$-cutting of the arrangement $\mathcal{A}(L)$ in the following standard manner. Draw a random sample R of $O(r \log r)$ lines of L, form the arrangement $\mathcal{A}(R)$ and triangulate its cells using vertical decomposition. We denote the resulting triangulated arrangement as $\mathcal{A}^{\|}(R)$. This produces $O(r^2 \log^2 r) = O^*(r^2)$ cells, and we may assume (since this occurs with high probability) that each cell is crossed by at most n_B/r lines of L (see [5, 9] for further details).

Consider a pair $(\delta, C) \in H_0$, where the touching between δ and $\partial^- C$ occurs at some cell τ of $\mathcal{A}^{\|}(R)$. In this case δ crosses τ (or has an endpoint inside τ), and σ_C either intersects τ or lies above τ (i.e., within the common x-span of C and τ, σ_C lies fully above τ). For technical reasons, we classify the arcs δ that cross τ as being either *short*, if δ does not intersect the top edge of τ, or *tall*, if δ intersects the top edge. Let A_τ^s be the set of short arcs in τ, and A_τ^t the set of tall arcs in τ.

The next lemma shows that the overall number of short arcs, over all cells τ, is small.

LEMMA 2.3. $\sum_\tau |A_\tau^s| = O(r^2 \log^3 r + |A^*| \log r)$.

Proof: We may ignore pairs (δ, τ), where δ ends inside τ; there are at most $2|A^*|$ such pairs. Construct a segment tree $\mathcal{T}$ on the x-projections of the cells of $\mathcal{A}^{\|}(R)$. Consider a node v of the tree, let Ξ_v denote the set of cells stored at v, and let I_v denote the x-span of v. The cells in Ξ_v are linearly ordered in the y-direction, in the sense that for each $x_0 \in I_v$ the vertical line $x = x_0$ crosses all of them in a fixed order; see Figure 5(b).

In each cell τ, there are at most two (either tall or short) arcs, whose x-spans *overlap*, but not contained in, I_v (the first intersects the vertical line through the left endpoint of I_v, and the second intersects the vertical line through its right endpoint), for a total of $O(r^2 \log^3 r)$ such arcs, over all cells τ and all nodes v of $\mathcal{T}$.

We thus continue the analysis for those (short) arcs δ of A^*, whose x-span is *contained* in I_v. There is at most one cell $\tau \in \Xi_v$ such that $\delta \in A_\tau^s$; see Figure 5(c). The number of nodes v at which δ has this property is $O(\log r)$, because I_v contains the x-coordinate of an endpoint (actually, both endpoints) of δ. Hence, the contribution of arcs δ as above to $\sum_\tau |A_\tau^s|$ is $O(|A^*| \log r)$. Combining this with the previous bound completes the proof of the lemma. □

Remark: 1) The fact that the arcs δ have pairwise openly disjoint x-projections is crucial for the bound that we obtain in Lemma 2.3. The decomposition of (a cover of) H_0 that we construct is a variant of the decomposition obtained in [4]; however, the analysis in [4] does not exploit the special structure of the arcs δ, and results in a suboptimal bound. 2) An individual arc δ may cross $\Omega(r)$ cells τ, each of whose top boundary is disjoint from δ. However, Lemma 2.3 shows that the overall number of these crossings, summed over all arcs δ, is relatively small.

Each cell τ for which $|A_\tau^s| > \frac{|A^*|}{r^2}$ is split, by vertical lines, into subcells, such that each subcell τ' satisfies $|A_{\tau'}^s| \le \frac{|A^*|}{r^2}$; the number of cells is still $O^*(r^2)$.

Fix a (new) cell τ, and form the complete bipartite graph $A_\tau^s \times \mathcal{C}_\tau^*$, where $\mathcal{C}_\tau^*$ consists of all sets $C \in B$ such that ℓ_C passes above τ. We associate the interval I_τ (the x-span of τ) with this graph. Since r is a constant, we have

$$\sum_\tau (|A_\tau^s| + |\mathcal{C}_\tau^*|) = O^*(m_A + n_B)$$

(where the constant of proportionality depends on r).

We next claim that the overall number of boundary touchings on the boundary of the union, occurring within a cell τ and involving a tall arc in τ, summed over all cells τ, is

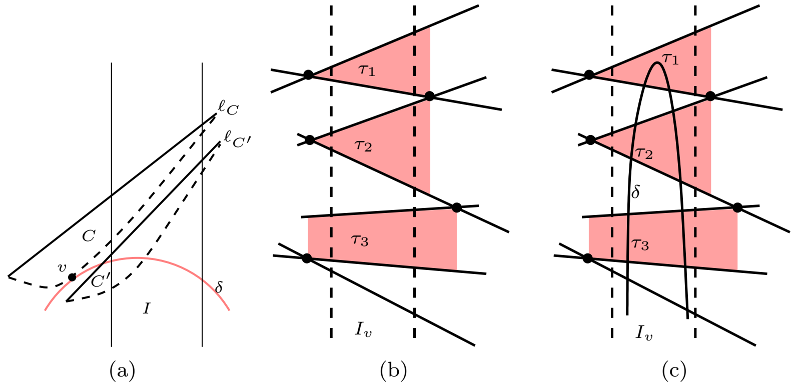

Figure 5: (a) The boundary touching v, formed by the lower boundary of C and δ, and lying outside the interval I, can be "hidden" from E_B^- by the lower boundary of another $C' \in B$. (b)–(c) The cells τ_1, τ_2, τ_3, cross the x-span I_v of v from left to right. (b) Any vertical line in I_v crosses all these cells in the same order. (c) The arc δ crosses τ_1, τ_2, τ_3, where τ_1 is the unique cell whose top boundary edge is not crossed by δ; δ is short there and tall at τ_2, τ_3.

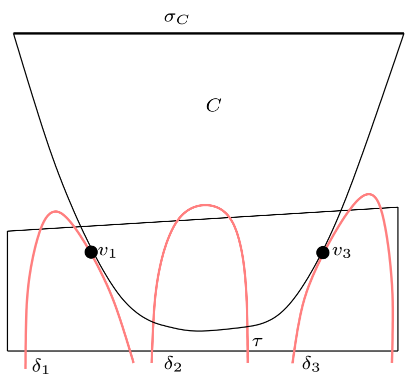

Figure 6: $\partial^- C$ touches δ_1 and δ_3 inside τ at v_1 and v_3, respectively, and cannot touch the intermediate tall arc δ_2.

only linear in n_B (and thus we need not provide a compact representation for these pairs). Indeed, let τ be a cell of $\mathcal{A}^{\|}(R)$, and let ℓ_C be the line containing the spine σ_C of a set C that intersects τ or passes fully above τ. We claim that there are at most two tall arcs in τ that touch $\partial^- C$ at a point that lies on ∂U. Indeed, suppose, to the contrary, that there are three such arcs $\delta_1, \delta_2, \delta_3$, which appear on E_A^+ in that order (from left to right). Consider the two respective boundary touchings that $\partial^- C$ forms with δ_1, δ_3, at two respective points v_1, v_3 inside τ. Then, due to the convexity of C, its portion between v_1 and v_3 lies below the top edge of τ, and δ_2 lies below that portion, so it cannot be tall in τ, a contradiction that establishes the claim; see Figure 6. Thus the overall number of regular vertices of the above kind is $O(n_B)$, as asserted.

We thus conclude that the overall number of boundary touchings involving both short and tall arcs, in all subcases considered so far, is $O^*(m_A + n_B)$.

We continue the construction recursively, within each cell τ, with A_τ^s and the subset $\mathcal{C}_\tau$ of those $C \in B$ whose line ℓ_C crosses τ. We have $|A_\tau^s| \le \frac{|A^*|}{r^2} = \frac{m_A}{r^2}$, $|\mathcal{C}_\tau| \le \frac{n_B}{r}$. However, the next stage of the recursion is performed in the *dual* plane, and proceeds as follows. For each resulting cell τ, map A_τ^s and $\mathcal{C}_\tau$ to the dual plane. For each $C \in \mathcal{C}_\tau$, we map ℓ_C to a dual point ℓ_C^*, and each arc δ in A_τ^s is mapped to a convex x-monotone curve δ^*, which is the locus of all points dual to lines that are tangent to δ (possibly at one of its endpoints) and pass above δ (see [4] and [6] for further details). Thus a line ℓ_C lies above an arc δ if and only if the dual point ℓ_C^* lies above δ^*. Each pair of dual arcs δ_1^*, δ_2^* intersect each other exactly once, since any such intersection point is the dual of a common tangent to δ_1, δ_2 that passes above both of them, and since δ_1, δ_2 are two convex curves that have disjoint x-spans, there is exactly one such common tangent. We now construct (for each cell τ obtained at the preceding step) a $(1/r)$-cutting of the arrangement of the dual arcs δ^*, obtaining $O^*(r^2)$ subcells, each of which is crossed by at most $\frac{|A_\tau^s|}{r} \le \frac{m_A}{r^3}$ dual arcs δ^*, and contains at most $\frac{|\mathcal{C}_\tau|}{r^2} \le \frac{n_B}{r^3}$ dual points ℓ_C^*; see [3] for details.

As above, we construct, for each subcell τ' of this cutting, a complete bipartite graph that connects the dual points in τ' to the dual arcs that pass fully below that subcell. Again, since r is a constant, the sum of the sizes of the vertex sets of these graphs is $O(m_A + n_B)$. We are thus left with $O^*(r^4)$ subproblems, each involving at most $\frac{m_A}{r^3}$ arcs δ of A^*, and at most $\frac{n_B}{r^3}$ sets in B. We now process each subproblem recursively, going back to the primal plane, and keep alternating in this manner, until we reach subproblems in which either $m_A^2 < n_B$, or $n_B^2 < m_A$. In the former (resp., latter) case, we continue the recursive construction *only* in the dual (resp., primal) plane, and stop as soon as one of m_A, n_B becomes smaller than r, in which case we output the complete bipartite graph $A_\tau^s \times \mathcal{C}_\tau$ involving the input sets to the subproblem. Note that in the bottom of the recurrence the boundary touchings are not necessarily obtained on the lower envelope of the boundary sets in $\mathcal{C}_\tau$, and thus the bound on their number in this particular case is $|A_\tau^s| \cdot |\mathcal{C}_\tau| = O(|A_\tau^s| + |\mathcal{C}_\tau|)$, where the constant of proportionality depends on r.

The preceding arguments imply that the union of all the bi-cliques constructed by this procedure, including the interactions with tall arcs and other "leftover" pairs detected by the decomposition, covers H_0. Indeed, for each such pair $(\delta, C) \in H_0$, the line ℓ_C containing the spine σ_C of C lies fully above δ. Our procedure detects all such pairs (δ, C) either (i) at the bottom of the recurrence, in which case all these pairs are reported in a brute force manner, or (ii) at a recursive step, performed in the primal plane and involving a cell τ in which the boundary touching appears, such that ℓ_C lies above τ and δ is short in τ, or (iii) at a recursive step, performed in the dual plane and involving a cell τ', such that ℓ_C^* lies inside τ' and δ^* passes fully below it.

Let $R(m_A, n_B)$ denote the maximum number of boundary touchings on the boundary of the union, that arise at a recursive step involving m_A arcs δ and n_B sets C, as above, and which are formed between one of the arcs δ and the bottom boundary of one of the sets C. As argued above, the number of such bichromatic touchings, that arise for any of the complete bipartite graphs generated at this stage, is nearly-linear in the sizes of the vertex sets of that graph. Hence R satisfies the following recurrence:

$$R(m_A, n_B) \le \begin{cases} O^*(m_A + n_B) + O^*(r^4) R\left(\frac{m_A}{r^3}, \frac{n_B}{r^3}\right), \\ \qquad \text{if } m_A^2 \ge n_B \ge \sqrt{m_A}, \\ O^*(m_A + n_B) + O^*(r^2) R\left(\frac{m_A}{r}, \frac{n_B}{r^2}\right), \\ \qquad \text{if } n_B > m_A^2, \\ O^*(m_A + n_B) + O^*(r^2) R\left(\frac{m_A}{r^2}, \frac{n_B}{r}\right), \\ \qquad \text{if } m_A > n_B^2, \\ O^*(m_A + n_B), \\ \qquad \text{if } \min\{m_A, n_B\} < r. \end{cases}$$

It is then easy to see, using induction on m_A and n_B, that the solution of this recurrence is

$$R(m_A, n_B) = O^*(m_A^{2/3} n_B^{2/3} + m_A + n_B) = \qquad (3)$$

$$O^*(n_A^{2/3} n_B^{2/3} + n_A + n_B).$$

Summing these bounds over all bi-cliques $A \times B$ of the first decomposition phase, and using the bound in (1), the upper bound of Theorem 2.1 follows.

Lower bounds. We introduce here a construction given in [13]. Construct a system of n lines and n points with $\Theta(n^{4/3})$ incidences between them (see, e.g., [14]). Map each line to a long and thin rectangle, and each point to a small disk, in such a way that, for each pair of a point p incident to a line ℓ, the disk into which p is mapped slightly penetrates the rectangle into which ℓ is mapped, and all the intersections between the boundaries of the disks and the rectangles lie on the boundary of their union. Clearly, $s = 4$ in this construction. Hence, we obtain a collection of $2n$ convex regions, each pair of whose boundaries intersect in at most four points, which have $\Theta(n^{4/3})$ regular vertices on the boundary of their union. This completes the proof of Theorem 2.1. □

Open problems. A major open problem is to extend the bound to the case where the sets in $\mathcal{C}$ are not convex. A natural case to study is where the sets in $\mathcal{C}$ are x-monotone, (i.e., each of the lower and upper portions of ∂C is an x-monotone curve), and, for each set $C \in \mathcal{C}$, the spine σ_C is contained in C, and each pair of boundaries intersect in a constant number, s, of points. In an earlier version of the paper, we obtained the upper bound $O^*(n^{(3s+1)/(2s+1)})$ for this case, which interpolates between the old bound $O^*(n^{3/2})$ of [4] and the new bound derived above. We tend to conjecture that the new bound $O^*(n^{4/3})$ also holds in this extended scenario, provided that each set in $\mathcal{C}$ has a constant description complexity[2].

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[2]That is, each set is defined as a Boolean combination of a constant number of polynomial equalities and inequalities of constant maximum degree.