On the Boundary of the Union of Planar Convex Sets*

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Abstract

We give two alternative proofs leading to different generalizations of the following theorem of [1]. Given n convex sets in the plane, such that the boundaries of each pair of sets cross at most twice, then the boundary of their union consists of at most $6n-12$ arcs. (An *arc* is a connected piece of the boundary of one of the sets.) In the generalizations we allow pairs of boundaries to cross more than twice.

1 Introduction

Let $\mathcal{C}$ be a collection of $n \geq 3$ non-degenerate convex sets (bodies) in the plane, any two of which have at most a finite number of boundary points in common. Assume for simplicity that the sets are in *general position*, i.e., no two boundary curves are tangent to each other, and no three pass through the same point. If two members of $\mathcal{C}$ have exactly two boundary points in common, then these points are called *regular vertices* of the arrangement $\mathcal{A}(\mathcal{C})$. All other intersection points of the boundary curves are said to be *irregular*. See Figure 1.

Let $U = \cup\mathcal{C}$ denote the union of all members of $\mathcal{C}$. Let R and I denote the set of regular and irregular vertices of $\mathcal{A}(\mathcal{C})$, respectively, lying on ∂U, the boundary of U. Further, put $V = R \cup I$. If the sets in $\mathcal{C}$ are bounded then $|V|$ is equal to the number of arcs that compose ∂U.

It was shown in [1] that if any two members of $\mathcal{C}$ have at most two boundary points in common (i.e., if there are no irregular vertices), then $|R| = |V| \leq 6n-12$, and this bound is tight in the worst case. In Section 2 of this note, we generalize this result as follows.

Theorem 1 *With the above notation, for any collection of $n \geq 3$ non-degenerate convex sets in general position in the plane satisfying the above assumptions, we have*

$$|R| \leq 2|I| + 6n - 12.$$

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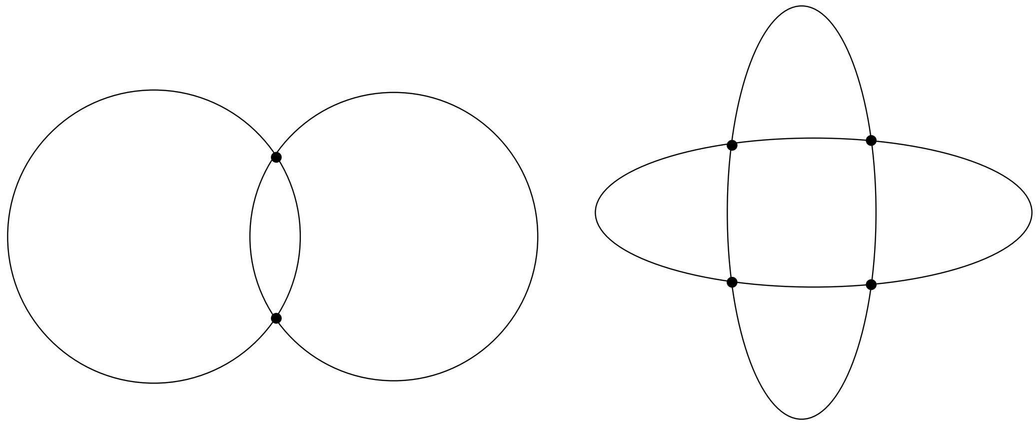

Figure 1: Regular vertices are shown on the left, and irregular vertices on the right

2. Formulation of Theorem 1: For any collection of $n \geq 3$ convex sets in the plane, we have...

Actually, in [1] the members of $\mathcal{C}$ were not required to be convex, and it is very likely that Theorem 1 also generalizes to that case.

Whitesides and Zhao [4] introduced the following definition. A collection of closed Jordan curves is called *k-admissible* if no two curves touch each other, any two curves intersect in at most k points, and the interior of no curve disconnects the interior of another. Clearly, we can restrict our attention to the case when k is even. In Section 3, we give a new proof of the following result of [4], which provides yet another generalization of the above mentioned theorem of [1].

Theorem 2 *The number of vertices on the boundary of the union of the interiors of $n \geq 3$ Jordan curves that form a k-admissible family, is at most $k(3n-6)$; this bound is tight in the worst case.*

The methods used here are quite different from those used in [1, 4].

2 Proof of Theorem 1

Preliminaries. We can assume without loss of generality that every member of $\mathcal{C}$ is bounded and that its boundary is smooth. It is sufficient to establish the theorem in the case when $U = \cup\mathcal{C}$ is connected; otherwise, arguing for each component of U separately, we obtain the stronger inequality $|R| \leq 2|I| + 6n - 12k_{\geq 3} - 10k_2 - 6k_1$, where k_1 (resp. k_2, $k_{\geq 3}$) is the number of connected components of U formed by one (resp. two, at least three) sets of $\mathcal{C}$.

A connected component H of the complement of U is called a *hole*. Let $V(H)$ denote the set of vertices along the boundary of a hole H. These vertices divide the boundary of H into $|V(H)|$ arcs, which form a set denoted by $\Gamma(H)$. The set of all arcs composing ∂U will be denoted by $\Gamma_{\text{ext}} = \cup_H \Gamma(H)$. Note that every bounded hole has at least three vertices. The unique unbounded hole may have fewer vertices (zero or two), but then $|V| \leq 2$. We may therefore assume that every hole has at least three vertices, so the number of holes is at most $|V|/3$.

Orient the boundary of every $c \in \mathcal{C}$ in the counter-clockwise direction. Accordingly, every (unit) tangent vector to c will be oriented so that c lies on its left-hand side.

Consider now two sets $c, c' \in \mathcal{C}$ whose boundaries intersect in exactly two points v and v'. (These are *regular* vertices of the arrangement.) Then $c \cap c'$ is a lens-like region, whose boundary is a counter-clockwise oriented closed curve $\xi_{cc'}$, with the two 'breakpoints' (nonsmooth points) v and v'. Denote the turning angles of (the tangents to) $\xi_{cc'}$ at v and v' by $a(v)$ and $a(v')$, respectively. (Note that $a(v)$, $a(v')$ are always positive.) See Figure 2.

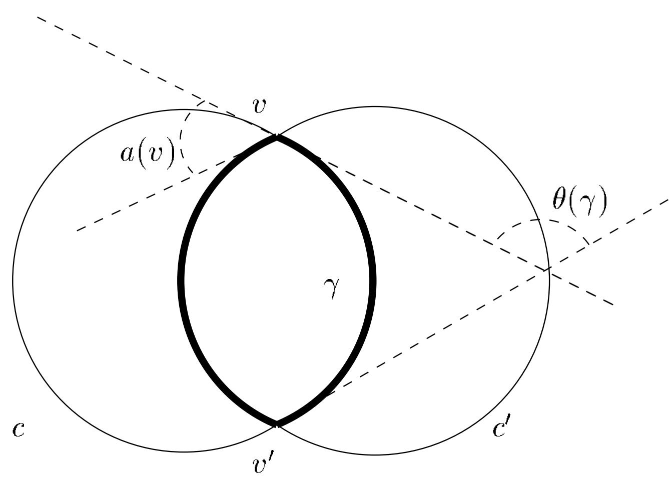

Figure 2: Two sets c, c' intersecting regularly, and the curve $\xi_{cc'}$. Also shown are the turning angle $\theta(\gamma)$ along the curve γ and the turning angle $a(v)$ at the vertex v.

Total turning angles of piecewise smooth curves. Let ξ be an oriented continuous curve in the plane. If at some point w of ξ, there is no unique tangent line, then w is called a *breakpoint*. We say that ξ is *piecewise smooth*, if it has finitely many breakpoints, and every piece of ξ between two consecutive breakpoints is differentiable (including at its endpoints).

Define the *total turning angle* $\theta(\xi)$ of a piecewise smooth, oriented curve ξ, as follows. If necessary, subdivide ξ into smaller differentiable oriented arcs $\xi_1, \ldots, \xi_m$, such that each ξ_i is smooth and any two tangents to the same arc ξ_i, oriented according to the orientation of the curve, differ in their orientations by less than π. Let $-\pi < \theta(\xi_i) < +\pi$ be the smaller angle from the tangent vector at the starting point of ξ_i to the tangent vector at the endpoint of ξ_i, taken with positive sign if the change is counter-clockwise and with negative sign otherwise (see, e.g., Figure 2). At each point w_i separating two pieces, ξ_i and ξ_{i+1}, let $\theta(w_i)$ be the smaller angle from the tangent to ξ_i at w_i to the tangent to ξ_{i+1} at w_i, with positive sign if and only if it is counter-clockwise. If w_i is not a breakpoint, then, by construction, $\theta(w_i) = 0$. Finally, let the total turning angle $\theta(\xi)$ be defined as the sum of $\theta(\xi_i)$ over all pieces ξ_i plus the sum of $\theta(w_i)$ over all vertices w_i. Evidently, this definition of the turning angle is independent of the particular subdivision of ξ. $\theta(\xi_i)$ and $\theta(w_i)$ are called, respectively, the *turning angle* of ξ along the arc ξ_i and at the point w_i.

The following lemma summarizes the elementary properties of the total turning angle. We omit the trivial proof.

Lemma 3 Let ξ be a piecewise smooth, oriented curve in the plane with total turning angle $\theta(\xi)$.

(i) If ξ is a closed curve, then $\theta(\xi)$ is an integer multiple of 2π.

(ii) If ξ is a counter-clockwise (resp. clockwise) oriented closed curve which does not intersect itself, then $\theta(\xi) = 2\pi$ (resp. -2π).

(iii) If ξ intersects itself at a point w, then it can be decomposed into two piecewise smooth, oriented curves, ξ' and ξ", having the common breakpoint w. (If ξ is a closed curve, then so are ξ' and ξ"; if ξ is open, then one of the two parts is open and the other is closed.) In both cases, we have

$$\theta(\xi) = \theta(\xi') + \theta(\xi").$$

We refer to the last equality as the *additivity property* of the total turning angle.

Turning angles along holes of the union. Notice that the orientation of the boundary of any *bounded* hole H of U is clockwise, and the orientation of the boundary of the unique *unbounded* hole is counter-clockwise. At any regular vertex v on the boundary of any hole, the turning angle of the boundary is $-a(v)$. Thus, Lemma 3 (ii) implies that for any fixed bounded hole H,

$$\sum_{v \in V(H)} (-a(v)) + \sum_{\gamma \in \Gamma(H)} \theta(\gamma) = -2\pi,$$

and, for the unique unbounded hole, the left-hand side is equal to 2π. Adding all these equations, multiplying by -1 and ignoring terms $a(v)$ for $v \in I$, we obtain

$$\sum_{v \in R} a(v) - \sum_{\gamma \in \Gamma_{\text{ext}}} \theta(\gamma) \leq 2\pi(|H| - 1) - 2\pi = 2\pi|H| - 4\pi \leq \frac{2\pi(|R| + |I|)}{3} - 4\pi. \quad (1)$$

Let Γ_{int} denote the collection of maximal boundary arcs of the sets in $\mathcal{C}$, oriented as above, that are contained in the interior of U. In the next section, we establish the following lemma.

Lemma 4

$$\sum_{v \in R} (\pi - a(v)) \leq \sum_{\gamma \in \Gamma_{\text{int}}} \theta(\gamma). \quad (2)$$

It is easy to see that Lemma 4 implies Theorem 1. Indeed, the right-hand side of (2) is equal to $2\pi n - \sum_{\gamma \in \Gamma_{\text{ext}}} \theta(\gamma)$. Summing up (1) and (2), we obtain

$$\pi|R| \leq 2\pi n + \frac{2\pi(|R| + |I|)}{3} - 4\pi,$$

which yields that $|R| \leq 2|I| + 6n - 12$, as asserted. □

3 Proof of Lemma 4

Let Γ^R denote the subset of those arcs in $\Gamma_{\rm int}$ that have at least one regular endpoint. The union of Γ^R is decomposed into a collection of oriented cycles and paths; the vertices (breakpoints) of the cycles and the internal vertices of the paths belong to R, and the endpoints of the paths belong to I. In the next two subsections (A and B), we prove that the total turning angle of *each* of these cycles and paths is at least $k\pi$, where k is the number of *all* vertices of a cycle or the number of *internal* vertices of a path. In subsection C, we show how Lemma 4 follows from this fact.

A: The case of a cycle. Let $\zeta = v_0 v_1 \cdots v_k$ $(v_k = v_0)$ be one of these oriented cycles, with vertices $v_0, v_1, \ldots, v_{k-1} \in R$. Let γ_i denote the oriented arc along ζ connecting v_{i-1} to v_i, and let c_i be the set in $\mathcal{C}$ whose boundary contains γ_i, for $i = 1, \ldots, k$.

We first consider the simple case $k = 2$. In this case, v_0 and v_1 are the two (regular) intersections of ∂c_1 and ∂c_2, and ζ is the convex curve $\xi_{c_1 c_2}$ defined above (see Figure 2). Clearly, $\theta(\zeta) = 2\pi$, as claimed.

Next suppose that $k \geq 3$. We traverse ζ from v_0, and consider the tangents to ζ, oriented in accordance with the orientation of ζ (so that the sets they are tangent to lie on their left). By construction, as we follow these tangents, they keep turning in the counter-clockwise (positive) direction, and this also holds at each vertex of ζ. See Figure 3.

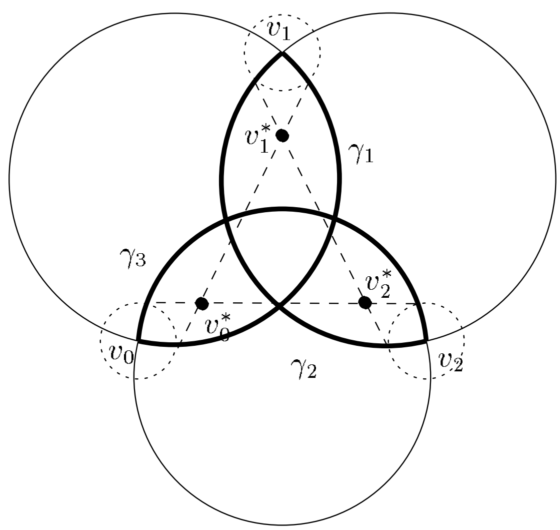

Figure 3: Illustrating the proof of Lemma 4 for a cycle of Γ^R

For each $i = 1, \ldots, k$, choose a very small $\varepsilon > 0$, and draw a circle of radius ε around each vertex v_i. Let v_i^- and v_i^+ denote an intersection points of this circle with γ_i and γ_{i+1}, respectively (with $\gamma_{k+1} = \gamma_1$). Let ζ' denote the closed curve obtained from ζ by replacing the portion of γ_i between v_{i-1}^+ and v_i^- by a straight-line segment, for every i. Clearly, the total turning angle of ζ' is equal to the total turning angle of ζ. See Figure 3.

We claim that ζ' can be decomposed into k positively (i.e., counter-clockwise) oriented loops at the vertices v_i and an oriented closed polygon $\zeta^* = v_0^* v_1^* \cdots v_k^*$; $(v_k^* = v_0^*)$. This follows from the fact that v_{i-1} and v_{i+1}, the other endpoints of the arcs γ_i and γ_{i+1}, lie on different sides of the line connecting v_i and the other regular intersection point v_i' of the boundaries of c_i and c_{i+1}. (Since $k \geq 3$, v_i' lies in the interior of the union, and γ_{i-1}, γ_i cross each other at that point.) Consequently, if ε is sufficiently small, then the segments $v_{i-1}^+ v_i^-$ and $v_i^+ v_{i+1}^-$ must cross each other in a small neighborhood of v_i, at a point denoted by v_i^*. The i-th loop of ζ' is its portion that starts and ends at v_i^*. Again, see Figure 3. Thus, by the additivity the turning angle,

$$\theta(\zeta') = k(2\pi) + \theta(\zeta^*).$$

By definition, at each vertex of ζ^*, the absolute value of the turning angle of ζ^* is at most π (and the turning angle along its edges is 0). Consequently, $\theta(\zeta) = \theta(\zeta') \geq k\pi$. (Actually, by Lemma 3(i), the total turning angle of ζ must be a multiple of 2π, so $\theta(\zeta) \geq 2\lceil k/2 \rceil \pi$ is also true. This is indeed the case shown in Figure 3: the total turning angle of ζ is $4\pi = 2\lceil 3/2 \rceil \pi$.)

B: The case of a path. Consider now a path $\zeta = v_0 v_1 \cdots v_k v_{k+1}$ with irregular endpoints and regular internal vertices. Let γ_i, c_i, for $i = 1, \ldots, k+1$, and v_i^-, v_i^+, for $i = 1, \ldots, k$, denote the same entities as for cycles (the previous case). We also put $v_0^+ = v_0$ and $v_{k+1}^- = v_{k+1}$. In exactly the same way as before, we construct a curve ζ' from ζ by replacing with a straight-line segment the portion of γ_i between v_{i-1}^+ and v_i^- , for every $i = 1, \ldots, k+1$. We have that $\theta(\zeta) \geq \theta(\zeta')$ (we turn more along γ_1 from v_0 to v_1^- than by going straight from v_0 to v_1^- and then turning at v_1^- until we are tangent to γ_1, and similarly at the other end of ζ; see Figure 4). Now, arguing as in the case of cycles, ζ' is decomposed into k positively oriented

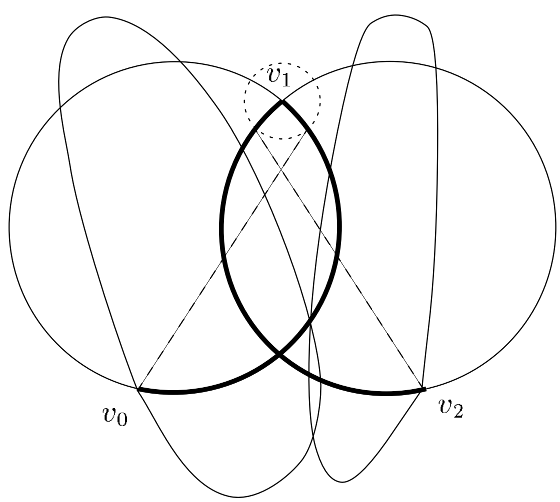

Figure 4: Illustrating the proof of Lemma 4 for a path of Γ^R

loops and a polygonal path ζ^*. Again, the additivity of the turning angle implies that

$$\theta(\zeta) \geq \theta(\zeta') = k(2\pi) + \theta(\zeta^*).$$

Since at each internal vertex of ζ^*, the turning angle is between $-\pi$ and $+\pi$, we have that the total turning angle of ζ is at least $k\pi$.

C: Putting it together. If ζ is a cycle, its total turning angle is, in the above notations,

$$\sum_{i=1}^{k} a(v_i) + \sum_{i=1}^{k} \theta(\gamma_i) \geq k\pi,$$

which implies that

$$\sum_{i=1}^{k} (\pi - a(v_i)) \leq \sum_{i=1}^{k} \theta(\gamma_i).$$

If ζ is a path, its total turning angle is, in the above notations,

$$\sum_{i=1}^{k} a(v_i) + \sum_{i=1}^{k+1} \theta(\gamma_i) \geq k\pi,$$

which implies that

$$\sum_{i=1}^{k} (\pi - a(v_i)) \leq \sum_{i=1}^{k+1} \theta(\gamma_i).$$

We now add these inequalities, over all cycles and paths composing Γ^R, and obtain

$$\sum_{v \in R} (\pi - a(v)) \leq \sum_{\gamma \in \Gamma^R} \theta(\gamma) \leq \sum_{\gamma \in \Gamma_{\rm int}} \theta(\gamma),$$

as asserted. □

4 Remarks

(A) In [1], we proved that $|R| \leq 6n - 12$, under the assumption that *all* vertices of $\mathcal{A}(\mathcal{C})$ are regular. Theorem 1 shows that the same bound holds with the weaker assumption that there are no irregular vertices on the boundary of $U = \cup\mathcal{C}$. (Recall, however, that the result in [1] does not require, as we do, that all members of $\mathcal{C}$ be convex.)

(B) Suppose that any two members of $\mathcal{C}$ have at most s (a constant number) of boundary points in common. How large can $|R|$ be? One can show that, even for $s = 4$, the maximum possible value of $|R|$ can be $\Omega(n^{4/3})$. To see this, take a set P of n points and a set L of n lines, so that there are $\Theta(n^{4/3})$ incidences between P and L (see [3, Chap. 11]). Replace each point in P by a disk of radius ϵ, for some sufficiently small $\epsilon > 0$, and replace each line $\ell \in L$ by a long rectangle whose width is ϵ and whose long bottom edge is parallel to ℓ, lying above ℓ, and at distance $\epsilon' < \epsilon$ from it. One can show that, for an appropriate choice of ϵ and ϵ', the number of intersections between any disk and any rectangle is at most two, that each incidence between a point of P and a line of L corresponds to an intersecting pair of a disk and a rectangle, and that each intersection point between such a pair lies on the boundary of the union. Hence, we have a collection of $2n$ disks and rectangles satisfying $|R| = \Theta(n^{4/3})$. Is this construction asymptotically best possible?

(C) It is not hard to see that the coefficient 2 of the term $|I|$ in Theorem 1 cannot be replaced by any smaller constant. To see this, take n copies of a regular n-gon, slightly rotated around their common center, and, for each original vertex, clip the batch of its copies with a small rectangle. This creates $2n^2$ regular vertices on the boundary of the union of the resulting collection of $2n$ convex sets. On the other hand, $|I|$ is about n^2. We also note that if $I \neq \emptyset$ then the bound in Theorem 1 is not tight, because we have ignored in (1) all terms $a(v)$ for $v \in I$, so we cannot have equality any more.

5 Proof of Theorem 2

Assume without loss of generality that every curve c has a point p_c that belongs to the boundary of U, the union of the interiors of all family members. Let q be one of the (at most k) intersection points of two curves, c and c'. Connect p_c to $p_{c'}$ by an arc ('edge'), going first from p_c to q in clockwise direction around c, and then following the boundary of c' in counterclockwise direction to $p_{c'}$. For each pair c, c' of family members that contribute an intersection point q to the boundary of U, construct such an edge that connects p_c to $p_{c'}$ via q, but do this for only one such point q. The two pieces an edge consists of are called *half-edges.* It is easy to show that any two half-edges not incident to the same point p_c intersect an even number of times. Thus, these edges form a graph drawing with the property that any two edges not incident to the same vertex p_c intersect an even number of times. This implies that the underlying graph is planar (see [2, Cor. 3.1]), and, since it has no multiple edges, the number of its edges is at most $3n - 6$. The total number of vertices along the boundary of U is obviously at most k times larger than that. To see that the bound is tight, use the same construction as in [1], but replace each pair of intersection points of a pair of boundaries by k consecutive intersections, all lying on the boundary of the union; refer to [1] for more details. $\square$

References

[1] K. Kedem, R. Livne, J. Pach, and M. Sharir, On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles, *Discrete Comput. Geom.* **1** (1986), 59–71.

[2] L. Lovász, J. Pach and M. Szegedy, On Conway's thrackle conjecture, *Proc. 11th ACM Symp. on Computational Geometry* (1995), 147–151.

[3] J. Pach and P.K. Agarwal, *Combinatorial Geometry,* J. Wiley-Interscience, New York, 1995.

[4] S. Whitesides and R. Zhao. K-admissible collections of Jordan curves and offsets of circular arc figures. *Technical Report SOCS 90.08*, McGill University, Montreal, 1990.