A Bipartite Analogue of Dilworth's Theorem for Multiple Partial Orders

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Abstract

Let r be a fixed positive integer. It is shown that, given any partial orders $>_1, \ldots, >_r$ on the same n-element set P, there exist disjoint subsets $A, B \subset P$, each with at least $n^{1-o(1)}$ elements, such that one of the following two conditions is satisfied: (1) there is an i $(1 \le i \le r)$ such that every element of A is larger than any element of B in the partial order $>_i$, or (2) no element of A is comparable with any element of B in any of the partial orders $>_1, \ldots, >_r$. As a corollary, we obtain that any family C of n convex compact sets in the plane has two disjoint subfamilies $A, B \subset C$, each with at least $n^{1-o(1)}$ members, such that either every member of A intersects all members of B, or no member of A intersects any member of B.

1 Introduction

A *chain* in a partially ordered set is a set of pairwise comparable elements and an *antichain* is a set of pairwise incomparable elements. Dilworth's celebrated theorem [4,13] implies that every partially ordered set on n elements contains a chain of length ℓ or an antichain of length $\lceil \frac{n}{\ell} \rceil$. Consequently, one can always find a chain or an antichain of length

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$\lceil\sqrt{n}\rceil$. Dilworth's theorem has motivated a great deal of research [14,22] in combinatorics and has several applications in combinatorial geometry [1,5,9,16,19,21], theoretical computer science [3,17], and set theory [8].

To prove Ramsey-type results on intersection patterns of convex bodies, Larman et al. [16] and Pach and Törőcsik [19] introduced *four* partial orders $<_1, <_2, <_3, <_4$ on the family of all convex bodies in the plane such that any two disjoint bodies were comparable with respect to at least one of them, but no two intersecting ones were. Applying Dilworth's theorem four times, we obtain that any family of n plane convex bodies has at least $n^{1/5}$ members that form a chain with respect to *some* i or an antichain with respect to *all* i $(1 \leq i \leq 4)$. In the first case, these sets are pairwise disjoint, in the second case pairwise intersecting. That is, we have

Theorem 1 *[16] Any family of n plane convex bodies has at least $n^{1/5}$ members that are either pairwise disjoint or pairwise intersecting.*

On the other hand, Károlyi et al. [15] constructed families of n convex sets (straight-line segments) in the plane with no subfamily consisting of more than than $n^{\log 4/\log 27} \approx n^{.41}$ pairwise intersecting or pairwise disjoint members. The above theorem remains true for *vertically convex* bodies, that is, for compact connected sets in the plane with the property that any straight line parallel to the y-axis of the coordinate system intersects it in an interval (which may be empty or may consist of one point). In particular, any x-monotone arc, that is, the graph of any continuous function defined on a subinterval of the x-axis, is vertically convex.

For a partially ordered set $(P, >)$, we write $a \perp b$ if a and b are incomparable. For subsets A and B of P, we write $A > B$ if $a > b$ for all $a \in A$ and $b \in B$. Likewise, we write $A \perp B$ if $a \perp b$ for all $a \in A$ and $b \in B$.

It was suggested by Erdős et al. [7] that under certain restrictions much stronger Ramsey-type results may hold if instead of large homogeneous subsets, we want to find large homogeneous (i.e., complete or empty) *bipartite* patterns. Indeed, for partially ordered sets, the first author proved the following bipartite analogue of Dilworth's theorem.

Theorem 2 *[10] Every n-element partially ordered set $(P, >)$ has two subsets $A, B \subset P$ with $|A| = |B| \geq \frac{n}{4\log_2 n}$ such that $A > B$ or $A \perp B$, provided that n is sufficiently large. This result is tight up to a constant factor.*

The same statement with $\frac{\sqrt{n}}{2}$ in place of $\frac{n}{4\log_2 n}$ immediately follows from Dilworth's theorem. Throughout this paper, all logarithms are of *base 2*.

In Section 2, we establish a generalization of the last theorem to multiple partial orders. We write $a \perp_i b$ to denote that a and b are *incomparable* by partial order $>_i$. Accordingly, for any subsets $A, B \subset P$, we write $A \perp_i B$ if $a \perp_i b$ for all $a \in A$ and for all $b \in B$.

Theorem 3 *Let r be a fixed positive integer, and let $>_1, \ldots, >_r$ be partial orders on an n-element set P. Then there are two disjoint subsets $A, B \subset P$, each with at least $\frac{n}{2^{(1+o(1))(\log\log n)^r}}$ elements, such that either $A >_i B$ for at least one i, or $A \perp_i B$ for all $1 \leq i \leq r$.*

Note that a straightforward, repeated application of Dilworth's theorem establishes the existence of two much smaller subsets $A, B \subset P$ with the above properties ($|A|, |B| \geq$

$\lfloor \frac{n^{1/(r+1)}}{2} \rfloor$).

Our next result shows that Theorem 3 is not very far from best possible. It will be proved in Section 3.

Theorem 4 *Let r be a positive integer, and let $0 < \epsilon < 1$. There is a constant $C(r, \epsilon) > 0$ such that for all sufficiently large positive integers n, there are r partial orders $<_1, \ldots, <_r$ on an n-element set P with the following properties:*

1. $<_1, \ldots, <_r$ have a common linear extension.

2. For any $v \in P$ and for any i $(1 \leq i \leq r)$, the number of elements in P comparable with v in the partial order $<_i$ is at most n^ϵ.

3. For any pair of disjoint subsets $A, B \subset P$, each with at least $C(r,\epsilon)n\frac{(\log\log n)^{r-1}}{(\log n)^r}$ elements, there exist $x \in A$, $y \in B$, and i $(1 \leq i \leq r)$ such that $x <_i y$ or $y <_i x$.

Applying Theorem 3 to the $r = 4$ partial orders on the family of plane convex bodies, introduced by Larman et al. [16] and Pach an Törőcsik [19], we immediately obtain the following

Theorem 5 *Any collection of n vertically convex compact sets in the plane has two disjoint subcollections, A and B, each with at least $\frac{n}{2^{(1+o(1))(\log\log n)^4}}$ members such that either every member of A intersects all members of B or every member of A is disjoint from all members of B.*

Based on the construction in [10], Pach and Tóth [18] showed that there is a collection of n vertically convex sets (x-monotone arcs) in the plane, which contains no subcollections A and B with the above properties such that $|A|, |B| \geq c\frac{n}{\log n}$, where c is a constant. This is a simple consequence of the second statement of Theorem 2 and the next lemma. Together they show that Theorem 5 is also not far from being best possible.

Lemma 6 *The elements of every partially ordered set $(\{1, 2, \ldots, n\}, \prec)$ can be represented by continuous real functions $f_1, f_2, \ldots, f_n$ defined on the interval $[0, 1]$ such that $f_i(x) < f_j(x)$ for every x if and only if $i \prec j$ $(i \neq j)$.*

In Section 4, we give a very short proof of Lemma 6 and make some concluding remarks.

2 Proof of Theorem 3

We need a simple technical lemma.

Lemma 7 *Let $S = S_1 \cup \ldots \cup S_m$ be a partition of a set S with $|S_i| = \ell$ for $1 \leq i \leq m$, and let A and B be disjoint subsets of S of the same size. For $1 \leq i \leq m$, let $A_i = A \cap S_i$ and $B_i = B \cap S_i$. Then there is a partition of the set $\{1, \ldots, m\}$ into two parts, I_1 and I_2, such that*

$$\sum_{i \in I_1} |A_i| \geq \frac{|A| - \ell}{2} \text{ and } \sum_{i \in I_2} |B_i| \geq \frac{|A| - \ell}{2}.$$

Proof: Define two subsets J_1 and J_2 of the index set by $J_1 = \{i : |A_i| \geq |B_i|\}$ and $J_2 = \{i : |B_i| \geq |A_i|\}$. Since

$$\sum_{i \in J_1} |A_i| + \sum_{i \in J_2} |B_i| \geq |A|,$$

we have that $\sum_{i \in J_1} |A_i| \geq \frac{|A|}{2}$ or $\sum_{i \in J_2} |B_i| \geq \frac{|A|}{2}$ holds. Suppose without loss of generality that $\sum_{i \in J_1} |A_i| \geq \frac{|A|}{2}$. Since each A_i has at most ℓ elements, there exists a subset $I_1 \subset J_1$ satisfying

$$\frac{|A| - \ell}{2} \leq \sum_{i \in I_1} |A_i| < \frac{|A| + \ell}{2}.$$

Letting $I_2 := \{1, 2, \ldots, m\} \setminus I_1$, we have

$$\sum_{i \in I_2} |B_i| = |A| - \sum_{i \in I_1} |B_i| \geq |A| - \sum_{i \in I_1} |A_i| > \frac{|A| - \ell}{2},$$

which proves the lemma. □

Now we are in a position to formulate a general statement for "monotone" families of graphs.

Theorem 8 *Let F_1, …, F_r be families of graphs that are closed by taking induced subgraphs, and let $f_i(n)$ be a monotonically increasing function such that for every graph $G \in F_i$ on n vertices, either G or the complement of G contains a complete bipartite subgraph with at least $\frac{n}{f_i(n)}$ vertices in each of its classes ($i = 1, \ldots, r$).*

Given any graphs $G_1 \in F_1, \ldots, G_r \in F_r$ on the same vertex set V, there exist disjoint subsets $V_1, V_2 \subset V$ with $\min(|V_1|, |V_2|) \geq \frac{n}{f(n)}$, where

$$\log f(n) = \prod_{i=1}^{r} \left(\lceil \log f_i(n) \rceil + 2 \right),$$

such that one of the following two conditions is satisfied: (1) there is an i ($1 \leq i \leq r$) such that in G_i every element of V_1 is adjacent to every element of V_2, or (2) in $G_1 \cup \ldots \cup G_r$, no element of V_1 is adjacent to any element of V_2.

Proof: For $r = 1$, the theorem is trivially true. The general statement follows from the special case $r = 2$ by a straightforward induction argument.

Let $k = 2 + \lceil \log f_2(n) \rceil$. By recursively iterating the definition of $f_1(n)$ on k levels, we obtain that either G_1 contains a complete bipartite graph $K_{m,m}$ with $m \geq \frac{n}{f_1(n)^k}$ (in which case we are done), or there are 2^k subsets $S_1, \ldots, S_{2^k}$ of V with $|S_i| = \lceil \frac{n}{f_1(n)^k} \rceil$ for $1 \leq i \leq 2^k$ such that there are no edges in G_1 between S_i and S_j for $1 \leq i < j \leq 2^k$. Let $S = S_1 \cup \ldots \cup S_{2^k}$. We are also done in the case when $K_{m,m}$ with $m \geq \frac{|S|}{f_2(n)}$ is a subgraph of the vertices induced by S in G_2, since

$$\frac{|S|}{f_2(n)} \geq \frac{2^k n}{f_2(n) f_1(n)^k} \geq \frac{n}{f_1(n)^k} \geq \frac{n}{2^{\log f_1(n) (\lceil \log f_2(n) \rceil + 2)}}.$$

Thus, we may assume that there are disjoint subsets A and B of S with $|A| = |B| > \frac{|S|}{f_2(n)}$ such that no vertex of A is adjacent to a vertex of B in G_2. Let $A_i = A \cap S_i$ and $B_i = B \cap S_i$ for $1 \leq i \leq 2^k$. Applying Lemma 7, we obtain a partition of $\{1, \ldots, 2^k\}$ into subsets I_1 and I_2 such that

$$\min\left(\sum_{i \in I_1} |A_i|, \sum_{i \in I_2} |B_i|\right) > \frac{|A| - |S_1|}{2} \geq \frac{|S|}{2f_2(n)} - \frac{|S_1|}{2} \geq |S_1| \geq \frac{n}{f_1(n)^k},$$

and no element of $V_1 = \bigcup_{i \in I_1} A_i$ is adjacent to any element of $V_2 = \bigcup_{i \in I_2} B_i$ in G_i for $i \in \{1, 2\}$. To complete the proof, we note that

$$\log_2\left(\frac{n}{\min(|V_1|, |V_2|)}\right) \geq k \log_2 f_1(n) = (\lceil \log_2 f_2(n) \rceil + 2) \log_2 f_1(n).$$

□

Now we are in a position to complete the proof of Theorem 3. One can associate with any set partially ordered set $(P, <)$ a *comparability graph*, whose vertex set is P and two vertices are connected by an edge if and only if one is larger than the other in the ordering.

Apply Theorem 8 to the families F_i of comparability graphs with respect to the partial orderings $<_i$ defined on all subsets of the underlying set P $(i = 1, \ldots, r)$. In view of Theorem 2, these families of graphs satisfy the conditions in the theorem with $f_i(n) = 4 \log n$. Thus, we can conclude that there are two disjoint subsets $A, B \subset P$, each with at least $\frac{n}{2^{(1+o(1))(\log \log n)^r}}$ elements, such that either there exists an i $(1 \leq i \leq r)$ such that every element of A is comparable to all elements of B with respect to $<_i$, or no element of A is comparable to any element of B with respect to any partial ordering $<_i$. In the latter case, we are done. In the former case, it is enough to refer to the following simple observation from [10].

Lemma 9 *Suppose that the comparability graph of a partially ordered set $(P, <)$ contains a complete bipartite subgraph with m vertices in each of its classes. Then there are two subsets $A, B \subset P$ with $|A| = |B| \geq \frac{m}{2}$ such that $A > B$.*

3 Construction

The aim of this section is to prove Theorem 4. The proof is by induction on r, based on Lemma 10 with appropriately chosen parameters.

The height $h(x)$ of an element x of a poset P is the length of the longest chain with largest element x. For a subset S of a partially ordered set $(P, <)$, let

$$D(S, h) = \{p : p \in P, h(p) \geq h \text{ and } p \leq s \text{ for at least one element } s \in S\},$$

and

$$U(S, h) = \{p : p \in P, h(p) \leq h \text{ and } p \geq s \text{ for at least one element } s \in S\}.$$

For a positive integer a and for a graph G on the vertex set $V = \{1, \dots, m\}$, define the poset $P(a, G)$ on the ground set $\{(j, l)\colon 1 \le j \le a \text{ and } 1 \le l \le m\}$ by setting $(j_1, l_1) < (j_2, l_2)$ whenever $j_2 = j_1 + 1$ and (l_1, l_2) is an edge of G. Let $P_j(a, G) = \{(j, l)\colon 1 \le l \le m\}$, that is, $P_j(a, G)$ is the set of elements of $P(a, G)$ of height j.

The *neighborhood* $N(S)$ of a set S of vertices consists of those vertices that are adjacent to at least one vertex in S. A graph $G = (V, E)$ is an *ϵ-expander* if for every subset $S \in V$ with $|S| \le \frac{|V|}{2}$ satisfies $|N(S)| \ge (1 + \epsilon)|S|$.

Lemma 10 *Let a, d, and Δ be positive integers and $G = (V, E)$ be a δ-expander graph with $|V| > 2d$, maximum degree Δ, and such that for any two subsets $V_1, V_2 \subset V$ of size at least d, there is an edge between a vertex of V_1 and a vertex of V_2.*

Then every element of $P(a, G)$ is comparable with at most $\frac{\Delta^a - 1}{\Delta - 1}$ other elements of $P(a, G)$, and if A and B are subsets of $P(a, G)$ such that $|A| = |B| > \frac{4(1+\delta)}{\delta} d$ and every element of A is incomparable with every element of B, then there is a j such that

$$|A \cap P_j(a, G)| \ge \frac{|A|}{4} \;\; \textit{and} \;\; |B \cap P_j(a, G)| \ge \frac{|B|}{4}.$$

Proof: The proof of the fact that every element of $P(a, G)$ is comparable with at most $\frac{\Delta^a - 1}{\Delta - 1}$ elements follows by a straightforward counting argument that can be found in [10].

Suppose that A and B are subsets of $P(a, G)$ such that $|A| = |B| > \frac{4(1+\delta)}{\delta} d$ and every element of A is incomparable with every element of B. We note that there is a j_0 and there are subsets $A' \subset A$ and $B' \subset B$ with $|A'| = |B'| \ge \frac{|B|}{2}$ such that either $h(x) \ge j_0 \ge h(y)$ for all $x \in A'$ and $y \in B'$, or $h(x) \le j_0 \le h(y)$ for all $x \in A'$ and $y \in B'$. Without loss of generality, we may assume that $h(x) \le j_0 \le h(y)$ for all $x \in A'$ and $y \in B'$.

Assume for contradiction that $|A' \cap P_{j_0}(a, G)| < \frac{|A|}{4}$ or $|B' \cap P_{j_0}(a, G)| < \frac{|B|}{4}$. Without loss of generality, we may assume that $|B' \cap P_{j_0}(a, G)| < \frac{|B|}{4}$.

Since every element of A is incomparable with all elements of B, we know that every element of $D(B, j_0 + 1)$ is incomparable with every element of $U(A, j_0)$. Using the fact that G is a δ-expander, we have

$$|D(B, j_0 + 1) \cap P_h(a, G)| \ge \min((1 + \delta)|D(B, j_0 + 1) \cap P_{h+1}(a, G)|, \frac{|V|}{2}),$$

for every $h \ge j_0 + 1$. Therefore, either

$$\frac{|V|}{2} \le |P_{j_0+1}(a, G) \cap D(B, j_0 + 1)|$$

holds, or

$$\frac{|B|}{4} \le |B'| - |B' \cap P_{j_0}(a, G)| \le |D(B, j_0 + 1)| = \sum_{1 \le i \le a - j_0} |P_{j_0+i}(a, G) \cap D(B, j_0 + 1)|$$

$$\leq \sum_{1\leq i\leq a-j_0} (1+\delta)^{i-1}|P_{j_0+1}(a,G)\cap D(B,j_0+1)| \leq \frac{1+\delta}{\delta}|P_{j_0+1}(a,G)\cap D(B,j_0+1)|.$$

Similarly, we have

$$\frac{|V|}{2} \leq |P_{j_0+1}(a,G)\cap U(A,j_0)|,$$

or

$$\frac{|A|}{4} \leq \frac{1+\delta}{\delta}|P_{j_0+1}(a,G)\cap U(A,j_0)|.$$

Let $V_1 \subset V$ be defined by $V_1 = \{l : l \in V \text{ and } (j_0, l) \in U(A, j_0)\}$, and let $V_2 \subset V$ be defined by $V_2 = \{l : l \in V \text{ and } (j_0+1, l) \in D(B, j_0+1)\}$. Notice that there are no edges between elements of V_1 and elements of V_2 and $\min(V_1, V_2) \geq \min(\frac{|V|}{2}, \frac{\delta}{4(1+\delta)}|A|)$. This contradicts the assumptions that $|V| > 2d$, $|A| = |B| > \frac{4(1+\delta)}{\delta}d$, and there is an edge between any pair of subsets of V of size at least d. □

Proof of Theorem 4: We prove Theorem 4 by induction on r. The case $r = 1$ was handled in the paper [10]. Our induction hypothesis is that for $r > 1$ and $0 < \epsilon < 1$ there is a constant $C(r-1, \epsilon)$ such that for every sufficiently large positive integer m, there is a set V on m elements and there are $r-1$ partial orders $<_1, \ldots, <_{r-1}$ on V with the property that for any $i \leq r-1$, no element of V is comparable with m^ϵ other elements of V by $<_i$, and for any pair $A, B \subset V$ of disjoint subsets with $|A| = |B| > C(r-1, \epsilon)m(\log m)^{1-r}(\log\log m)^{r-2}$, there is an index $j \leq r-1$ and elements $x \in A$ and $y \in B$ such that either $x <_j y$ or $y <_j x$.

It is an easy computational exercise to show that $G(m, (\log m)^{2r}/m)$ has, with probability tending to 1 as $m \to \infty$, the following three properties:

(1) $G(m, (\log n)^{2r}/m)$ is a 1/2-expander graph.
(2) $G(m, (\log m)^{2r}/m)$ has maximum degree less than $(\log m)^{4r}$.
(3) For any pair A, B of disjoint subsets of the vertex set of $G(m, (\log m)^{2r}/m)$, each of cardinality greater than $m/(\log m)^r$, there is an edge between a vertex of A and a vertex of B.

Let n be a sufficiently large positive integer. Let $a = \lfloor \frac{\epsilon \log n}{4r \log\log n} \rfloor$ and $m = \lceil n/a \rceil$. Let $G = (V, E)$ be a graph on m vertices satisfying properties (1)-(3) above. Let P consist of the am elements of $P(a, G)$, and let $<_r$ be the partial order on $P(a, G)$. By Lemma 10, every element of $P(a, G)$ is comparable with at most $((\log m)^{4r})^a < n^\epsilon$ other elements of $P(a, G)$. Also by Lemma 10, if A and B are subsets of P with $|A| = |B| > \frac{20}{3}m/(\log m)^r$ such that every element of A is incomparable with every element of B with respect to $<_r$, then there is $j \leq a$ such that $|P_j(a, G) \cap A| \geq \frac{|A|}{4}$ and $|P_j(a, G) \cap B| \geq \frac{|B|}{4}$.

Let $\prec_1, \ldots, \prec_{r-1}$ be partial orders on the set V such that for each $i \leq r-1$, no element of V is comparable with m^ϵ other elements of V by partial order $\prec_i$, and for each pair A and B of disjoint subsets of V with $|A| = |B| > C(r-1, \epsilon)\frac{m(\log\log m)^{r-2}}{(\log m)^{r-1}}$, there is $j \leq r-1$ and there are elements $x \in A$ and $y \in B$ such that either $x \prec_j y$ or $y \prec_j x$.

Define the partial orders $<_1, \ldots, <_{r-1}$ on the set P by $(j_1, l_1) <_i (j_2, l_2)$ if and only if $j_1 = j_2$ and $l_1 \prec_i l_2$. So for each $i \leq r-1$, no element of P is comparable with $m^\epsilon < n^\epsilon$ other elements of P by $<_i$.

Let $C(r,\epsilon) = \frac{32r}{\epsilon} C(r-1,\epsilon)$. Assume for contradiction that there exist disjoint subsets $A, B \subset P$ with $|A| = |B| > C(r,\epsilon) n \frac{(\log\log n)^{r-1}}{(\log n)^r}$ such that every element of A is incomparable with every element of B by the partial orders $<_1, \ldots, <_r$. By Lemma 2 (as explained above), it follows that there are disjoint subsets $V_1, V_2 \subset V$ with $|V_1| = |V_2| \geq \frac{|A|}{4}$ such that every element of V_1 is incomparable with every element of V_2 by the partial orders $\prec_1, \ldots, \prec_{r-1}$. By the induction hypothesis, for n sufficiently large, we have

$$|V_1| \leq C(r-1,\epsilon) m \frac{(\log\log m)^{r-2}}{(\log m)^{r-1}} < C(r-1,\epsilon) \lceil n/a \rceil \frac{(\log\log n)^{r-2}}{(\log n)^{r-1}}$$

$$\leq \frac{8r}{\epsilon} C(r-1,\epsilon) n \frac{(\log\log n)^{r-1}}{(\log n)^r} \leq \frac{|A|}{4} \leq |V_1|,$$

a contradiction. □

4 Concluding remarks

Short proof of Lemma 6. Let $(\{1,2,\ldots,n\},\prec)$ be a partial order, and let Π denote the set consisting of all of its extensions $\pi(1) \prec \pi(2) \prec \ldots \prec \pi(n)$ to a total order. Clearly, every element of Π is a permutation of the numbers $1,2,\ldots,n$. Let $\pi_1, \pi_2, \ldots, \pi_m$ be an arbitrary labelling of the elements of Π. Assign to each π_k a distinct point x_k of the interval $[0,1]$, so that

$$0 = x_1 < x_2 < \ldots < x_m = 1.$$

For each i $(1 \leq i \leq n)$, define a continuous, piecewise linear function $f_i(x)$, as follows. For any k $(1 \leq k \leq m)$, set $f_i(x_k) = \pi_k^{-1}(i)$, and let $f_i(x)$ change linearly over the interval $[x_k, x_{k+1}]$ for $k < m$.

Obviously, whenever $i \prec j$ for some $i \neq j$, we have that $\pi_k^{-1}(i) \prec \pi_k^{-1}(j)$ for every k, and hence $f_i(x) < f_j(x)$ for all $x \in [0,1]$. On the other hand, if i and j are incomparable with respect to the ordering $\prec$, we find that there are indices k and k' $(1 \leq k \neq k' \leq m)$ such that $f_i(x_k) < f_j(x_k)$ and $f_i(x_k) > f_j(x_{k'})$, therefore, by continuity, the graphs of f_i and f_j must cross at least once in the interval $(x_k, x_{k'})$. This completes the proof of Lemma 6. □

This proof also works using any collection of linear extensions whose intersection is the partial order. The *dimension* of a partial order is the minimum number of linear extensions whose intersection is the partial order. It is clear then that the minimum number of breakpoints needed for the construction with polygonal paths is the dimension of the partial order.

Intersection graphs of curves. As was noted before, we cannot expect that there always exist sets A and B with at least a positive constant times n members, satisfying the properties required in Theorem 5. On the other hand, it is possible that this holds for families of convex bodies in the plane. To prove such a statement, one probably needs to explore the geometric structure of such families; a straightforward application of a combinatorial result for multiple partial orders will not suffice.

The first steps in this direction are made in a subsequent paper by the authors and C. Tóth [12]. By "filling out" every member of a family of vertically convex bodies in the plane with a sufficiently fine x-monotone curve, one can obtain a family of x-monotone curves with the same intersection graph. Thus, instead of intersection graphs vertically convex bodies, we may restrict our attention to intersection graphs of x-monotone curves. It is proved in [12] that there exists a positive constant c such that if G is the intersection graph of a family of n x-monotone curves in the plane, then either G contains a complete bipartite graph with at least $\frac{cn}{\log n}$ vertices in each of its classes, or the complement of G contains a complete bipartite graph with at least cn vertices in each of its classes. Hence, it follows that the same is true for intersection graphs of vertically convex bodies in the plane. This result is tight up to a constant factor [19].

An *arrangement of pseudosegments* is a family of continuous arcs in the plane such that any pair intersect at most once. In [12], Theorem 8 is applied to establish the following result: There is a constant $c > 0$ with the property that if G is the intersection graph of an arrangement of $n > 1$ pseudosegments, each of which crosses a fixed line precisely once, then either G or its complement contains a complete bipartite graph with at least cn vertices in each of its classes. (It is very likely that here the condition that every pseudosegment intersects a given line precisely once can be dropped.) The last result implies that there is a positive constant c such that the intersection graph of any arrangement of n pseudosegments contains a complete subgraph or an independent set of size at least n^c.

Higher dimensions. In *three-* and higher dimensional spaces, there are no nontrivial, general Ramsey-type theorems for families of convex bodies. This is due to the fact that *every* finite graph can be obtained as the intersection graph of convex bodies in $\mathbb{R}^3$ (see, e.g., Tietze [20]). However, for families of "fat" convex bodies, one can establish some nontrivial results of this type. A convex body S in $\mathbb{R}^d$ is called *K-fat* if there are d-dimensional balls B_1 and B_2 with $B_1 \subset S \subset B_2$ such that the ratio of the radius of B_2 to the radius of B_1 is at most K. In [11], it is proved that for every K and d, there is a positive constant $\epsilon = \epsilon(K, d)$ such that for any family of n K-fat convex bodies in $\mathbb{R}^d$, either there is a point contained in at least ϵn members of the family, or the complement of the intersection graph of these bodies has a complete bipartite graph with at least ϵn vertices in each of its classes. Consequently, there is a positive constant $c = c(K, d)$ such that the intersection graph of any arrangement of n K-fat convex bodies in $\mathbb{R}^d$ contains a complete subgraph or an independent set of size at least n^c.

Erdős-Hajnal-type results. We say that a class of graphs $\mathcal{G}$ has the *Erdős–Hajnal property* if it is closed under taking induced subgraphs and there exists an $\epsilon > 0$ such that every member $G \in \mathcal{G}$ has either a complete subgraph or an independent set of size at least $|V(G)|^\epsilon$. According to a well known conjecture of Erdős and Hajnal [6], for any graph H, the class of graphs containing no induced subgraph isomorphic to H has the Erdős–Hajnal property. The combinatorial core of the proof of Theorem 1 in [16] is the following simple statement: For every $\epsilon > 0$ and for every positive integer r, there is a $\delta = \delta(\epsilon, r) > 0$ such that, if $\mathcal{F}_i$ $(1 \le i \le r)$ are families of graphs that have the Erdős–Hajnal property with exponent

$\epsilon > 0$, then the class of all graphs that can be obtained as $\cup_{1\leq i\leq r}F_i$ for some $F_i \in \mathcal{F}_i$ $(1 \leq i \leq r)$, also has the Erdős–Hajnal property, with the exponent δ. Moreover, for any such graph $F = \cup_{1\leq i\leq r}F_i$, there is a subset $U \subset V(F)$ with $|U| \geq |V(F)|^{\delta}$ such that U induces either a complete subgraph in *some* F_i or an independent set in *all* F_is (and hence in F). It is easy to see that here $\delta(\epsilon, r)$ can be taken to be ϵ^r.

We say that a class of graphs has the *strong Erdős–Hajnal property* if it is closed under taking induced subgraphs and there exists an $\epsilon > 0$ such that for every member $G \in \mathcal{G}$ that has at least two vertices, either G or its complement $\overline{G}$ has a complete bipartite subgraph with at least $\epsilon|V(G)|$ vertices in each of its classes. It was shown in [2] that the strong Erdős–Hajnal property implies the Erdős–Hajnal property.

Theorem 8 has the following immediate corollary that can be regarded as a bipartite version of the above statement from [16].

Corollary 11 *For every $\epsilon > 0$ and for every positive integer r, there is a $\gamma = \gamma(\epsilon, r) > 0$ such that, if $\mathcal{F}_i$ $(1 \leq i \leq r)$ are families of graphs that have the strong Erdős–Hajnal property with parameter $\epsilon > 0$, then the class of all graphs that can be obtained as $\cup_{1\leq i\leq r}F_i$ for some $F_i \in \mathcal{F}_i$ $(1 \leq i \leq r)$, also has the Erdős–Hajnal property, with the parameter γ.*

Moreover, for any such graph $F = \cup_{1\leq i\leq r}F_i$, there are disjoint subsets $U_1, U_2 \subset V(F)$ with $|U_1|, |U_2| \geq \gamma|V(F)|$ such that either there is an index i $(1 \leq i \leq r)$ such that every vertex of U_1 is adjacent to every vertex of U_2 in F_i, or no vertex of U_1 is adjacent to any vertex of U_2 in F.

By Theorem 8, $\gamma(\epsilon, r)$ can be taken to be $2^{-\left(3+\log\frac{1}{\epsilon}\right)^r}$.

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