Conflict-free Colorings

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Abstract

A coloring of the elements of a planar point set P is said to be *conflict-free* if, for every closed disk D whose intersection with P is nonempty, there is a color that occurs in $D \cap P$ precisely once. We solve a problem of Even, Lotker, Ron, and Smorodinsky by showing that any conflict-free coloring of every set of n points in the plane uses at least $c \log n$ colors, for an absolute constant $c > 0$. Moreover, the same assertion is true for homothetic copies of any convex body D, in place of a disk.

1 Introduction

Motivated by a frequency assignment problem in cellular telephone networks, Even, Lotker, Ron, and Smorodinsky [ELRS02] studied the following question. Given a set P of n points in the plane, what is the smallest number of colors in a coloring of the elements of P with the property that any closed disk D with $D \cap P \neq \emptyset$ has an element whose color is not assigned to any other element of $D \cap P$. We refer to such a coloring as a *conflict-free* coloring of P with respect to disks.

In the specific application, the points correspond to *base stations* interconnected by a fixed backbone network. Each *client* continuously scans frequencies in search of a base station within its (circular) range with good reception. Once such a base station is found, the client establishes a radio link with it, using a frequency not shared by any other station within its range. Therefore, a conflict-free coloring of the points corresponds to an assignment of frequencies to the base stations, which enables every client to connect to a base station without interfering with the others.

Even et al. proved that any set of n points in the plane has a conflict-free coloring with $O(\log n)$ colors, and they exhibited an example showing that this bound cannot be improved. The aim of the present note is to show that in fact *any* set of n points requires at least constant times $\log n$ colors for a conflict-free coloring.

Theorem 1. *Every conflict-free coloring of every set of n points in the plane uses at least $\log_8 n$ colors.*

In Section 3, we show that a similar result holds for conflict-free colorings with respect to non-circular ranges (see Theorem 2), and we discuss some related questions.

2 Proof of Theorem 1

Throughout this section, we fix a set P of n points in the plane and a conflict-free coloring of P with k colors.

It is sufficient to establish the following

Lemma 2.1. *For any $0 \leq i < k$, there is a closed axis-parallel square $\boldsymbol{S}_i$ with circumscribing circle $\boldsymbol{C}_i$ such that*
(a) $|P \cap \boldsymbol{S}_i| \geq \frac{n}{8^i}$, and
(b) the elements of P belonging to the interior of $\boldsymbol{C}_i$ are colored with at most $k-i$ colors.

Applying the lemma with $i = k-1$, we obtain a circle $\boldsymbol{C}_{k-1}$ containing $m \geq \frac{n}{8^{k-1}} - 4$ points of P in its interior (not counting the corners of the inscribed square $\boldsymbol{S}_{k-1}$, which may belong to P). By (ii), all of these points must have the same color. Using the property that the coloring is conflict-free, we have that $m \leq 1$, so that $k - 1 \geq \log_8(n/5)$ and Theorem 1 follows.

It remains to prove Lemma 2.1. We use induction on i. For $i = 0$, let $\boldsymbol{S}_0$ be any axis-parallel closed square containing P. Suppose that, for some $0 \leq i \leq k-2$, we have already found a square $\boldsymbol{S}_i$ with circumscribing circle $\boldsymbol{C}_i$, satisfying the requirements of the lemma. Denote the vertices of $\boldsymbol{S}_i$ by V_1 (upper left), V_2 (upper right), V_3 (lower right), and V_4 (lower left). The coloring of P is conflict-free, so among the elements of P in the interior of $\boldsymbol{C}_i$ there is one with a unique color. Pick such a point and denote it by O.

We distinguish two cases.

Case A: $O \notin \boldsymbol{S}_i$.

For any $1 \leq j \leq 4$, define $\boldsymbol{S}^j = \boldsymbol{S}^j_i$ to be the *largest* axis-parallel closed square with circumscribing circle $\boldsymbol{C}^j = \boldsymbol{C}^j_i$ satisfying the following three conditions:

(i) $\boldsymbol{S}^j \subseteq \boldsymbol{S}_i$,
(ii) V_j is a vertex of $\boldsymbol{S}^j$,
(iii) O does not lie in the interior of $\boldsymbol{C}^j$.

Claim 2.2. *For any $1 \leq j \leq 4$,*
(a) circle $\boldsymbol{C}^j$ is inside $\boldsymbol{C}_i$,
(b) $\bigcup_{j=1}^4 \boldsymbol{S}^j = \boldsymbol{S}_i$.

Proof. Part (a) follows directly from the definition.

Suppose without loss of generality that O belongs to the part of the disk enclosed by $\boldsymbol{C}_i$ that is cut off by the segment V_1V_2 (see Fig. 1, left). For any square $\boldsymbol{S}$, let $l(\boldsymbol{S})$ denote its side length. Draw a vertical line through O, and let O' denote its intersection with V_1V_2.

The square $\overline{\boldsymbol{S}}^1$ with upper left corner V_1 and upper right corner O' and its circumscribing circle $\overline{\boldsymbol{C}}^1$ satisfy conditions (i)–(iii) with $j = 1$. Therefore, by the maximality of $\boldsymbol{S}^1$, we have $l(\boldsymbol{S}^1) \geq l(\overline{\boldsymbol{S}}^1) = V_1O'$. Similarly, we obtain $l(\boldsymbol{S}^2) \geq V_2O'$, whence

$$l(\boldsymbol{S}^1) + l(\boldsymbol{S}^2) \geq V_1V_2 = l(\boldsymbol{S}_i). \tag{1}$$

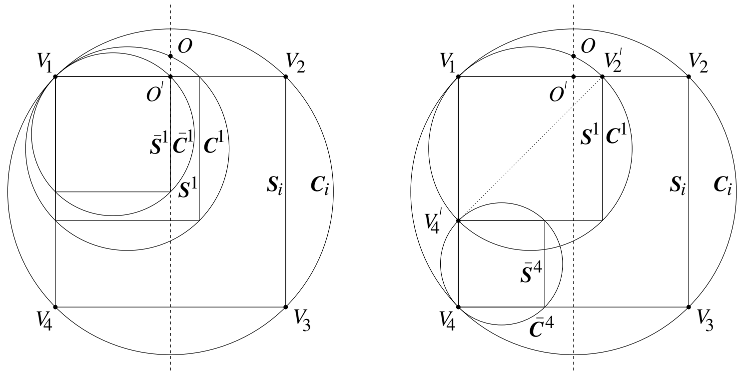

Fig. 1.

Let V_4' and V_2' denote the lower left and the upper right corners of $\boldsymbol{S}^1$, respectively. Consider the closed square $\overline{\boldsymbol{S}}^4$ with lower left corner V_4 and upper left corner V_4', and let $\overline{\boldsymbol{C}}^4$ denote its circumscribing circle (see Fig. 1, right). Notice that the line $V_4'V_2'$ is tangent to $\overline{\boldsymbol{C}}^4$ and it separates $\overline{\boldsymbol{C}}^4$ from O' and O. Therefore, $\overline{\boldsymbol{S}}^4$ and $\overline{\boldsymbol{C}}^4$ satisfy conditions (i)–(iii) with $j = 4$. Hence, by the maximality of $\boldsymbol{S}^4$, we have $l(\boldsymbol{S}^4) \geq l(\overline{\boldsymbol{S}}^4) = V_4V_4'$ so that

$$l(\boldsymbol{S}^1) + l(\boldsymbol{S}^4) \geq V_1V_4 = l(\boldsymbol{S}_i). \tag{2}$$

By symmetry, we also have

$$l(\boldsymbol{S}^2) + l(\boldsymbol{S}^3) \geq l(\boldsymbol{S}_i). \tag{3}$$

Now let $\widehat{\boldsymbol{S}}^4$ denote the closed square whose lower left corner is V_4 and whose size is the same as that of $\boldsymbol{S}^1$, and let $\widehat{\boldsymbol{C}}^4$ denote its circumscribing circle (see Fig. 2). Again, it is easy to check that $\widehat{\boldsymbol{S}}^4$ and $\widehat{\boldsymbol{C}}^4$ satisfy conditions (i)–(iii) with $j = 4$. Thus, we obtain $l(\boldsymbol{S}^4) \geq l(\widehat{\boldsymbol{S}}^4) = l(\boldsymbol{S}^1)$ and, analogously, $l(\boldsymbol{S}^3) \geq l(\boldsymbol{S}^2)$. Therefore, we have

$$l(\boldsymbol{S}^3) + l(\boldsymbol{S}^4) \geq l(\boldsymbol{S}^2) + l(\boldsymbol{S}^1) \geq l(\boldsymbol{S}_i). \tag{4}$$

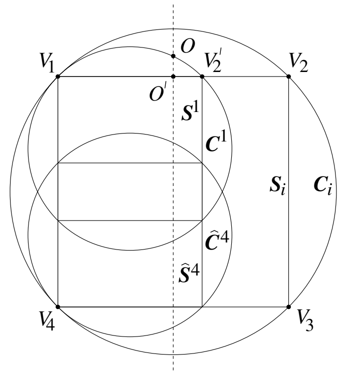

Fig. 2.

Equations (1)–(4) immediately imply part (b) of the claim. □

Case B: $O \in \boldsymbol{S}_i$.

For any $1 \le j \le 4$, define the closed square $\boldsymbol{S}^j$ with circumscribing circle $\boldsymbol{C}^j$, as in Case A. Furthermore, for any $5 \le j \le 8$, let $\boldsymbol{S}^j$ denote the *largest* axis-parallel closed square with circumscribing circle $\boldsymbol{C}^j$ satisfying the following three conditions:

(i) $\boldsymbol{S}^j \subseteq \boldsymbol{S}_i$;

(ii) O is the lower right corner of $\boldsymbol{S}^5$, the lower left corner of $\boldsymbol{S}^6$, the upper left corner of $\boldsymbol{S}^7$, and the upper right corner of $\boldsymbol{S}^8$;

(iii) O does not lie in the interior of $\boldsymbol{C}^j$.

Claim 2.3. *For any* $1 \le j \le 8$,
(a) circle $\boldsymbol{C}^j$ *is inside* $\boldsymbol{C}_i$,
(b) $\bigcup_{j=1}^{8} \boldsymbol{S}^j = \boldsymbol{S}_i$.

Proof. Part (a) follows from the fact that each $\boldsymbol{S}^j$ is contained in and can be obtained from $\boldsymbol{S}_i$ by shrinking it from one of its points q. Therefore, q must lie on or inside of $\boldsymbol{C}_i$, and the same shrinking transformation will take $\boldsymbol{C}_i$ into $\boldsymbol{C}^j$, proving that $\boldsymbol{C}^j \subset \boldsymbol{C}_i$.

Draw a vertical and a horizontal line through O, and denote their intersections with the four sides of $\boldsymbol{S}_i$ by V^{up}, V^{down}, V^{left}, and V^{right}. These two lines divide $\boldsymbol{S}_i$ into four rectangles. To prove part (b) of the claim, it is enough to show, by symmetry, that the upper left rectangle, $\boldsymbol{R} = V_1 V^{\mathrm{up}} O V^{\mathrm{left}}$ is covered by $\boldsymbol{S}^1 \cup \boldsymbol{S}^5$ (see Fig. 3, left).

Suppose without loss of generality that $V_1 V^{\mathrm{up}} \ge V^{\mathrm{up}} O$, so that $l(\boldsymbol{S}^5) = V^{\mathrm{up}} O$. Let V_1' denote the upper left corner of $\boldsymbol{S}^5$.

Assume first that $V_1 V^{\mathrm{left}} \ge V_1 V_1'$. Let $\overline{\boldsymbol{S}}^1$ be the closed square whose upper left and lower left corners are V_1 and V^{left}, resp., and let $\boldsymbol{C}$ denote its circumscribing circle. By our assumption, we have $\overline{\boldsymbol{S}}^1 \cup \boldsymbol{S}^5 = \boldsymbol{R}$. On

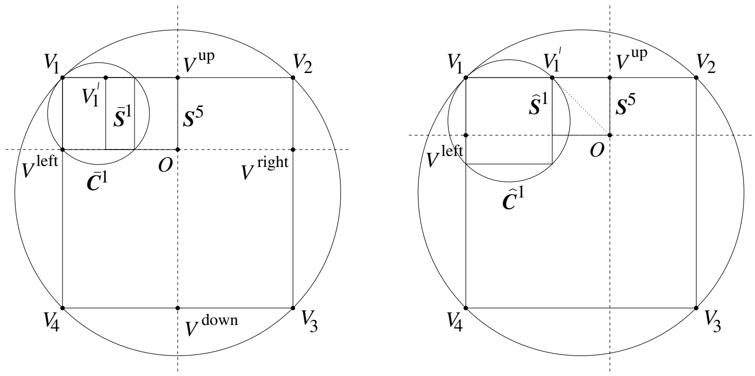

Fig. 3.

the other hand, $\overline{\boldsymbol{S}}^1$ and $\overline{\boldsymbol{C}}^1$ obviously satisfy conditions (i)–(iii) with $j = 1$. Thus, by the maximality of $\boldsymbol{S}^1$, we have $\boldsymbol{S}^1 \supseteq \overline{\boldsymbol{S}}^1$, yielding that $\boldsymbol{S}^1 \cup \boldsymbol{S}^5 \supseteq \boldsymbol{R}$.

Assume next that $V_1V^{\rm left} \le V_1V_1'$. Now let $\widehat{\boldsymbol{S}}^1$ denote the closed square with upper left corner V_1 and upper right corner is V_1', and let $\widehat{\boldsymbol{C}}^1$ be its circumscribing circle. Line $V_1'O$ is tangent to $\widehat{\boldsymbol{C}}^1$, so O cannot lie inside $\widehat{\boldsymbol{C}}^1$ (see Fig. 3, right). Therefore, by our assumption, $\widehat{\boldsymbol{S}}^1 \cup \boldsymbol{S}^5 \supseteq \boldsymbol{R}$. Since $\widehat{\boldsymbol{S}}^1$ and $\widehat{\boldsymbol{C}}^1$ satisfy conditions (i)–(iii) with $j = 1$, we obtain $\boldsymbol{S}^1 \supseteq \widehat{\boldsymbol{S}}^1$. Consequently, we also have $\boldsymbol{S}^1 \cup \boldsymbol{S}^5 \supseteq \boldsymbol{R}$. This completes the proof of part (b) of Claim 2.3. □

Now we are in a position to complete the induction step in the proof of Lemma 2.1. By the induction hypothesis, we have $|P \cap \boldsymbol{S}_i| \ge \frac{n}{8^i}$. In both cases (A and B), condition (iii) guarantees that, for every j, all points in the interior of $\boldsymbol{C}^j$ are colored with at most $k - i - 1$ colors, because the color of O cannot be used. On the other hand, by parts (b) of Claims 2.2 and 2.3, there *exists* a j $(1 \le j \le 8)$ such that $|P \cap \boldsymbol{S}^j| \ge |P \cap \boldsymbol{S}_i|/8 \ge \frac{n}{8^{i+1}}$. Setting $\boldsymbol{S}_{i+1} := \boldsymbol{S}^j$ and $\boldsymbol{C}_{i+1} := \boldsymbol{C}^j$, the assertion of the lemma follows for $i + 1$.

This concludes the proof of Lemma 2.1 and hence the theorem.

3 A generalization and concluding remarks

For any compact convex body D in the plane, a coloring of the elements of a point set P is said to be D*–conflict-free* if, for any *homothetic* (i.e., translated and similar) copy of D, whose intersection with P is nonempty, there is a color that occurs in $D \cap P$ precisely once.

Even et al. [ELRS02] extended their result on disks by showing that, for any given D, any set of n points permits a D–conflict-free coloring with

$O(\log n)$ colors. They gave an example of n points requiring $\Omega(\log n)$ colors. The argument presented in the previous section easily generalizes to this case.

Theorem 2. *For any compact convex body D, every D–conflict-free coloring of every set of n points in the plane uses at least $\log_8 n$ colors.*

Proof. (Sketch) Let X and Y be two points on the boundary of D at maximum distance from each other, i.e., let XY be a *diameter* of D. Let ℓ_1 and ℓ_2 be two lines parallel to XY such that the lengths of the segments $X_1Y_1 = D \cap \ell_1$ and $X_2Y_2 = D \cap \ell_2$ are equal to half of the length of XY. Applying a proper affine transformation to the plane (including D and the point set), the parallelogram $X_1Y_1X_2Y_2$ becomes an axis-parallel square. So, without loss of generality, we can assume that D has an inscribed square. Now one can repeat the proof of Theorem 1 with the only difference that, instead of axis-parallel squares with circumscribing circles, we have to use axis-parallel squares with circumscribing homothetic copies of D. $\Box$

In the same spirit, for any family of sets, $\mathcal{F}$, a coloring of a point set P is said to be $\mathcal{F}$*–conflict-free* or *conflict-free with respect to* $\mathcal{F}$ if, for every member $F \in \mathcal{F}$ whose intersection with P is nonempty, there is a color that appears in $F \cap \mathcal{F}$ precisely once.

It was pointed out in [HS02] that every set of n points in general position in the plane permits a conflict-free coloring using $O(\sqrt{n})$ colors, with respect to all axis-parallel closed rectangles. This can be slightly improved, as follows.

Proposition 3.1. *Every set of n points in general position in the plane permits a conflict-free coloring using $O(\sqrt{n \log\log n / \log n})$ colors, with respect to the family of all axis-parallel closed rectangles.*

Proof. Given a set P of n points in general position (i.e., no two of them have the same x-coordinate or y-coordinate), define a graph G on the vertex set P by connecting two points with an edge if and only if the smallest axis-parallel closed rectangle containing both of them has no element of P in its interior. It is easy to verify that G is "uncrowded:" it has no complete subgraph with 5 vertices.

We *claim* that G has an independent set of size $\Omega\left(\sqrt{n \log n / \log\log n}\right)$. For any $p \in P$, the vertical and horizontal lines passing through p divide the plane into four quadrants. Obviously, the neighbors of p lying in each of these quadrants form either a monotone increasing or a monotone decreasing subsequence, i.e., all slopes determined by their point pairs have the same sign. Taking every other element of each of these sequences, we obtain an independent set, so that the neighborhood of p can be decomposed into at most *eight* (in fact, at most *four*) independent sets. Therefore, if there exists a point $p \in P$ whose degree is at least $\Omega\left(\sqrt{n \log n / \log\log n}\right)$, we are done. Otherwise, the maximum degree D in G satisfies $D = O\left(\sqrt{n \log n}\right)$. Now we can apply an extension of a result of Ajtai, Komlós, and Szemerédi [AKS80] on uncrowded graphs, due to Shearer [S95]. This implies that G has an independent set of size $\Omega\left((n/D)(\log D/\log\log D)\right) = \Omega\left(\sqrt{n\log n/\log\log n}\right)$.

We follow the approach of [ELRS02] to argue that Proposition 3.1 is an easy consequence of the above *claim*. Pick an independent set $S_1 \subseteq P$ of size $\Omega\left(\sqrt{n\log n/\log\log n}\right)$ in G. Color all elements of S_1 with color 1, and use the claim to find a large independent set S_2 in the subgraph of G induced by $P - S_1$. Color all elements of S_2 with color 2. Continue like this until no points are left. The resulting coloring will meet the requirements of Proposition 3.2. $\square$

References

[AKS80] M. Ajtai, J. Komlós, and E. Szemerédi, A note on Ramsey numbers, *J. Combin. Theory Ser. A* **29** (1980), 354–360.

[ELRS02] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky, Conflict–free colorings of simple geometric regions with applications to frequency assignment in cellular networks, in: *Proceedings of 43rd Annual IEEE Symposium on the Foundations of Computer Science*, 691–700, 2002.

[HS02] S. Har-Peled and S. Smorodinsky, On conflict-free coloring of points and simple regions in the plane, manuscript.

[S95] J. B. Shearer, On the independence number of sparse graphs, *Random Structures and Algorithms* **7** (1995), 269–271.

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Acknowledgments

János Pach has been supported by NSF grant CR-00-98246, PSC-CUNY Research Award 63382-0032 and OTKA-T-032452. Géza Tóth has been supported by OTKA-T-038397, OTKA-T-032452 and the New York University Research Challenge Fund.

We are indebted to Shakhar Smorodinsky for bringing the above problems to our attention and for making many interesting remarks. He informed us that Proposition 3.1 was also proved by Noga Alon and Timothy Chan.