Colorings with only rainbow arithmetic progressions

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Abstract

If we want to color $1, 2, \ldots, n$ with the property that all 3-term arithmetic progressions are *rainbow* (that is, their elements receive 3 distinct colors), then, obviously, we need to use at least $n/2$ colors. Surprisingly, much fewer colors suffice if we are allowed to leave a negligible proportion of integers uncolored. Specifically, we prove that there exist $\alpha, \beta < 1$ such that for every n, there is a subset A of $\{1, 2, \ldots, n\}$ of size at least $n - n^{\alpha}$, the elements of which can be colored with n^{β} colors with the property that every 3-term arithmetic progression in A is rainbow. Moreover, β can be chosen to be arbitrarily small. Our result can be easily extended to k-term arithmetic progressions for any $k \geq 3$.

As a corollary, we obtain a simple proof of the following result of Alon, Moitra, and Sudakov, which can be used to design efficient communication protocols over shared directional multi-channels. There exist $\alpha', \beta' < 2$ such that for every n, there is a graph with n vertices and at least $\binom{n}{2} - n^{\alpha'}$ edges, whose edge set can be partitioned into at most $n^{\beta'}$ *induced matchings*.

Dedicated to the 80th birthday of Endre Szemerédi.

1 Introduction

Szemerédi's regularity lemma [14] started a new chapter in extremal combinatorics and in additive number theory. In particular, it was instrumental in proving a famous conjecture of Erdős and Turán, according to which, for every real number $\delta > 0$ and every integer $k > 0$, there exists a positive integer $n = n(\delta, k)$ such that every subset of $[n] = \{1, 2, \ldots, n\}$ that has at least δn elements contains an arithmetic progression of length k (in short, a k-*AP*); see [13]. The $k = 3$ special case of this theorem, originally proved by Roth [11], also follows from the celebrated *triangle removal lemma* [12], which is another direct consequence of the regularity lemma. It has several other closely related formulations and consequences:

1. If A is subset of $[n]$ with no 3-AP, then $|A| = o(n)$.
2. If G is a graph on n vertices whose edge set can be partitioned into n *induced* matchings, then $|E(G)| = o(n^2)$.

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3. If G is a graph on n vertices which has $o(n^3)$ triangles, then one can eliminate all triangles by removing $o(n^2)$ edges of G.

4. If H is a system of triples of $[n]$ such that every 6-element subset of $[n]$ contains at most 2 triples in H, then $|H| = o(n^2)$.

More precisely, the above statements apply to any infinite series of sets A, graphs G, and triple systems H, resp., where $n \to \infty$.

An old construction of Behrend [5] shows that there are 3-AP-free sets $A \subset [n]$ of size at least $ne^{-O(\sqrt{\log n})}$, so that 1 is not far from being tight. Ruzsa and Szemerédi [12] observed, that Behrend's construction can be used to show the existence of graphs G with n vertices and $|E(G)| \geq n^2 e^{-O(\sqrt{\log n})}$ edges that can be partitioned into n induced matchings. Hence, 2 is also nearly tight, and the same is true for 3 and 4.

Szemerédi's theorem on arithmetic progressions immediately implies van der Waerden's theorem [15]: For any integer $k \geq 3$, let $c_k(n)$ denote the minimum number of colors needed to color all elements of $[n]$ without creating a *monochromatic* k-AP. Then we have $\lim_{n\to\infty} c_k(n) = \infty$.

How many colors do we need if, instead of trying to avoid monochromatic k-term arithmetic progressions, we want to make sure that every k-term arithmetic progression is *rainbow*, that is, all of its elements receive distinct colors? For instance, it is easy to see that for $k = 3$, we need at least $n/2$ colors. Surprisingly, it turns out that much fewer colors suffice if we do not insist on coloring *all* elements of $[n]$. In particular, there is a subset of $A \subset [n]$ with $|A| = (1 - o(1))n$ whose elements can be colored by $n^{o(1)}$ colors with the property that all 3-term arithmetic progressions in A are rainbow.

More precisely, we prove the following result.

Theorem 1. *There exist $\alpha, \beta < 1$ with the following property. For every sufficiently large positive integer n, there is a set $A \subset [n]$ with $|A| \geq n - n^\alpha$ and a coloring of A with at most n^β colors such that every 3-term arithmetic progression in A is rainbow.*

Moreover, for every $\beta > 0$, we can choose $\alpha < 1$ satisfying the above conditions.

Theorem 1 can be used to construct graphs with n vertices and $(1 - o(1))\binom{n}{2}$ edges which can be partitioned into a small number of induced matchings. The first such constructions were found by Alon, Moitra, and Sudakov [2]. Theorem 1 easily implies the main result of [2], which is as follows.

Corollary 2. *There exist $\alpha', \beta' < 2$ with the following property. For every sufficiently large positive integer n, there is a graph with n vertices and at least $\binom{n}{2} - n^{\alpha'}$ edges that can be partitioned into $n^{\beta'}$ induced matchings.*

Moreover, for every $\beta' > 1$, we can choose $\alpha' < 2$ satisfying the above condition.

Dense graphs that can be partitioned into few induced matchings have been extensively studied, partially due to their applications in graph testing [1, 3, 4, 9] and testing monotonicity in posets [7]. The graphs satisfying the conditions in Corollary 2 can be used to design efficient communication protocols over shared directional multi-channels [6, 2]. Some other interesting graphs decomposable into large matchings were constructed and studied in [8].

Our proof of Theorem 1 is inspired by the construction of Behrend [5], but it also has a lot in common with one of the two constructions given by Alon, Moitra, and Sudakov [2]. Roughly, the idea of Behrend is to identify the elements of $[n]$ with a high dimensional grid $[C]^d$, in which we find a sphere passing through many grid points. These points will correspond to a dense 3-AP-free set in $[n]$. We proceed similarly, but instead of taking a sphere, we take a small neighborhood S of a sphere. If we choose the radii properly, it follows by standard concentration laws that almost all points of the grid $[C]^d$ are contained in S. On the other hand if 3 points form a 3-AP in S, then they must be close to each other. This observation can be explored to give a coloring of $S \cap [C]^d$ with the desired properties.

In Sections 2 and 3, we prove Theorem 1 and Corollary 2, respectively. In the last section, we indicate how to extend Theorem 1 to k-term arithmetic progressions for any $k \geq 3$; see Theorem 6.

2 Rainbow 3-AP's—Proof of Theorem 1

We start by setting a few parameters. Let C be a sufficiently large integer. Suppose for simplicity that $n = C^d$ for some integer d. The general case can be treated in a similar manner. In the sequel, log will stand for the natural logarithm.

Set $\epsilon = \frac{1}{C^3}$ and let $B = \{0, 1, \ldots, C-1\}^d$, so that $|B| = C^d$. We view B as a subset of the vector space $\mathbb{R}^d$ endowed with the Euclidean norm $|.|$. For $\mathbf{x} \in B$, let $\mathbf{x}(i) \in \{0, 1, \ldots, C-1\}$ denote the ith coordinate of $\mathbf{x}$, where $1 \leq i \leq d$. Clearly, the map $\phi : B \to [n]$ defined as

$$\phi(\mathbf{x}) = 1 + \sum_{i=1}^{d} \mathbf{x}(i) C^{i-1}$$

is a bijection.

Let $\mathbf{z}$ be an element chosen uniformly at random from the set B, and let $r = (\mathbb{E}[|\mathbf{z}|^2])^{1/2}$. We have

$$r^2 = \mathbb{E}[|\mathbf{z}|^2] = \sum_{i=1}^{d} \mathbb{E}[\mathbf{z}(i)^2] = \frac{d(C-1)(2C-1)}{6}.$$

Therefore,

$$\frac{dC^2}{6} < r^2 < \frac{dC^2}{3}.$$

Let A' consist of the set of all points in B that lie in the spherical shell between the spheres of radii $r(1-\epsilon)$ and $r(1+\epsilon)$ about the origin. That is, let

$$S = \{\mathbf{x} \in \mathbb{R}^d : r(1-\epsilon) \leq |\mathbf{x}| \leq r(1+\epsilon)\},$$

and let $A' = B \cap S$. Finally, set $A = \phi(A')$. Next we show, using standard concentration laws, that A' contains almost all elements of B and, hence, A contains almost all elements of $[n]$.

Claim 3. $|A| = |A'| \geq C^d(1 - 2e^{-\frac{1}{18}d\epsilon^2}) = n - 2n^{1-\frac{\epsilon^2}{18\log C}}$.

Proof. Note that $|\mathbf{z}|^2 = \sum_{i=1}^{d} \mathbf{z}(i)^2$ is the sum of d independent random variables taking values in $\{0, \ldots, (C-1)^2\}$. We have $r^2 = \mathbb{E}[|\mathbf{z}|^2] \le C^2 d$. On the other hand, if $\mathbf{x} \notin A'$, then $||\mathbf{x}|^2 - r^2| > \epsilon r^2 > (1/6)\epsilon d C^2$. Thus, by Hoeffding's inequality [10], we obtain

$$1 - \frac{|A'|}{C^d} \le \mathbb{P}\left[||\mathbf{z}|^2 - r^2| > (1/6)\epsilon d C^2\right] \le 2e^{-\frac{1}{18} d\epsilon^2} = 2n^{-\frac{\epsilon^2}{18 \log C}}.$$

□

Therefore, with the choice $\alpha = 1 - \frac{\epsilon^2}{30 \log C}$, we have $|A| \ge n - n^{\alpha}$, provided that n is sufficiently large.

It remains to define a coloring c of A with the desired properties. Using the bijection ϕ between B and $[n]$, this corresponds to a coloring of $A' \subset B$. We would like to guarantee that for every $a, b \in A$ with $a \neq b$ and $c(a) = c(b)$, we have $\frac{a+b}{2}$ and $2a - b \notin A$. (By swapping a and b, the latter condition also implies that $2b - a \notin A$.) Equivalently, we want that if $c(a) = c(b)$ and $\frac{a+b}{2}, 2a - b \in [n]$, then

$$\phi^{-1}\left(\frac{a+b}{2}\right) \quad \text{and} \quad \phi^{-1}(2a-b) \notin A'.$$

To achieve this, we would like to use the identities

$$\phi^{-1}\left(\frac{a+b}{2}\right) = \frac{\phi^{-1}(a) + \phi^{-1}(b)}{2} \quad \text{and} \quad \phi^{-1}(2a-b) = 2\phi^{-1}(a) - \phi^{-1}(b).$$

However, these equations hold if and only if

$$\frac{\phi^{-1}(a) + \phi^{-1}(b)}{2} \in B \quad \text{and} \quad 2\phi^{-1}(a) - \phi^{-1}(b) \in B,$$

respectively.

To overcome this problem, we first give an auxiliary coloring f of B such that if $f(\mathbf{x}) = f(\mathbf{y})$, then

$$\frac{\mathbf{x}+\mathbf{y}}{2} \quad \text{and} \quad 2\mathbf{x} - \mathbf{y} \in B.$$

We define f as follows. For any $\mathbf{x} \in B$, let $f(\mathbf{x}) = (a_1, \ldots, a_d, b_1, \ldots, b_d)$, where, for every $i \in [d]$, we have

$$a_i = \begin{cases} 0 & \text{if } \mathbf{x}(i) \text{ is even,} \\ 1 & \text{if } \mathbf{x}(i) \text{ is odd.} \end{cases}$$

and

$$b_i = \begin{cases} k & \text{if } \mathbf{x}(i) \le \frac{C}{2} \text{ and } 2^{k-1} - 1 \le \mathbf{x}(i) < 2^k - 1, \\ -k & \text{if } \mathbf{x}(i) > \frac{C}{2} \text{ and } 2^{k-1} - 1 \le C - 1 - \mathbf{x}(i) < 2^k - 1. \end{cases}$$

Then f uses at most $2^d (2\log_2 C)^d$ colors, and it is easy to verify that f satisfies the desired properties.

Next, we define a coloring g of A' such that any 3-AP in A' is rainbow. Then, the coloring (f, g) induces a coloring on A for which every 3-AP is rainbow. In order to define g, we need a simple geometric observation; see Figure 1.

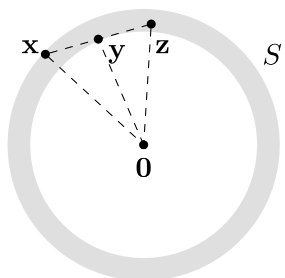

Figure 1: An illustration for Claim 4.

Claim 4. *If* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S$ *such that* $\mathbf{y} = \frac{\mathbf{x}+\mathbf{z}}{2}$, *then* $|\mathbf{x} - \mathbf{z}| \leq 4\sqrt{\epsilon} r$.

Proof. At least one of the angles $\mathbf{0yx}$ and $\mathbf{0yz}$ is at least $\frac{\pi}{2}$, see Fig. 1. Assume without loss of generality that $\mathbf{0yx}$ is such an angle. Then we have $|\mathbf{y}|^2 + |\mathbf{y} - \mathbf{x}|^2 \leq |\mathbf{x}|^2$. On the other hand, $|\mathbf{x}|^2 \leq (1+\epsilon)^2 r^2$ and $|\mathbf{y}|^2 \geq (1-\epsilon)^2 r^2$, so that we obtain

$$|\mathbf{x} - \mathbf{z}|^2 = 4|\mathbf{y} - \mathbf{x}|^2 \leq 4(|\mathbf{x}|^2 - |\mathbf{y}|^2) \leq 16\epsilon r^2.$$

□

Define a graph G on the vertex set A', as follows. Join $\mathbf{x}, \mathbf{y} \in A'$ by an edge if at least one of the 3 vectors $2\mathbf{x} - \mathbf{y}, \frac{\mathbf{x}+\mathbf{y}}{2}, 2\mathbf{y} - \mathbf{x}$ belongs to A'.

Claim 5. *Let* Δ *denote the maximum degree of the vertices of* G. *Then we have*

$$\Delta < 2^d C^{16\epsilon d C^2}.$$

Proof. Fix any $\mathbf{x} \in A'$. By Claim 4, every neighbor of $\mathbf{x}$ is at distance at most $4\sqrt{\epsilon} r < 4\sqrt{\epsilon}\sqrt{d} C$ from $\mathbf{x}$. If $|\mathbf{x} - \mathbf{y}| \leq 4\sqrt{\epsilon}\sqrt{d} C$ for some $\mathbf{y} \in A'$, then there are at most $16\epsilon d C^2$ indices $i \in [d]$ such that $\mathbf{x}(i) \neq \mathbf{y}(i)$. The number of vertices $\mathbf{y}$ with this property is smaller than $2^d C^{16\epsilon d C^2}$. Indeed, there are fewer than 2^d ways to choose the indices i for which $\mathbf{x}(i) \neq \mathbf{y}(i)$ and, for each such index i, there are fewer than C different choices for $\mathbf{y}(i)$. Therefore, we have

$$\Delta < 2^d C^{16\epsilon d C^2}.$$

□

It follows from Claim 5 that G has a proper coloring with at most $\Delta + 1$ colors. By the definition of G, if in such a coloring two elements are colored with the same color, then this pair is not contained in any 3-AP in A'.

In th end, we obtain the coloring (f, g) of A' with at most

$$(\Delta + 1) 2^d (2 \log_2 C)^d \leq (10 C^{16\epsilon C^2} \log_2 C)^d$$

colors such that every 3-AP in A' is rainbow. The coloring c on A induced by (f, g) has the same property.

Using that $\epsilon = \frac{1}{C^3}$, we have $D = 10C^{16\epsilon C^2}\log_2 C < C$, provided that C is sufficiently large. Letting $\beta = \log_C D$, the number of colors used by c is at most n^β.

Increasing C, β tends to zero. Thus, in view of Claim 3, we obtain that for every $\beta > 0$, there is a suitable positive $\alpha < 1$ which satisfies the conditions of Theorem 1. $\square$

3 Induced matchings—Proof of Corollary 2

Let $\gamma > 0$, $s = n^\gamma$ and $m = n^{1-\gamma}$ (for simplicity, we omit the use of floors and ceilings). Let V be a set of size n, and partition V into s sets $V_1, \ldots, V_s$ of size m. Let $\alpha, \beta < 1$ denote two constants meeting the requirements of Theorem 1. We will show that Corollary 2 is true with suitable constants $\alpha' = \max\{1 + \alpha - \alpha\gamma, 2 - \gamma\} + o(1)$ and $\beta' = 1 + \beta + \gamma - \beta\gamma + o(1)$, as $n \to \infty$. This illustrates that by chosing γ sufficiently small, we can guarantee that β' can be arbitrarily close to 1.

Let $A \subset [2m]$ be a set of size at least $2m - (2m)^\alpha$, and let c be a coloring of A with at most $(2m)^\beta$ colors such that every 3-AP in A is rainbow.

Construct a graph G on the vertex set V, as follows. Identify each V_i with the set $[m]$ and, for every $1 \le i < j \le s$ and $x \in V_i, y \in V_j$, connect x and y by an edge of G if and only if $x + y \in A$. If xy is an edge, color it with the color

$$c'(xy) = (i, j, x - y, c(x + y)).$$

Note that the same symbol x denotes a different vertex in each V_i. Also, the third coordinate of the color $c'(xy)$ can be negative, zero, or positive.

First, we show that each color class is an induced matching. In other words, we show that if $xy \neq uv$ are distinct edges of G such that $c'(xy) = c'(uv) = c'$, then xy and uv do not share a vertex and none of xu, xv, yu, yv can be an edge of G having color c'. The first two coordinates of the color $c'(xy) = c'(uv) = c'$ determine the pair of indices $(i, j), i < j$, such that both xy and uv run between V_i and V_j. Suppose without loss of generality that $x, u \in V_i$ and $y, v \in V_j$. If $x = u$, say, then $c'(xy) = c'(uv)$ implies that $x - y = u - v$, so that $y = v$, contradicting our assumption that xy and uv are distinct edges. Therefore, xy and uv cannot share a vertex. By definition, there is no edge between x and u, and there is no edge between y and v.

It remains to show that neither xv, nor yu can be an edge of color c'. Let $d = x - y = u - v$. Suppose, for example, that xv is an edge of color c'. Then $x + v \in A$, and we have

$$\frac{(x + y) + (u + v)}{2} = \frac{(2x - d) + (2v + d)}{2} = x + v.$$

Comparing the left-hand side and the right-hand side, it follows that $x + y, x + v, u + v$ are distinct numbers that form a 3-AP in A. However, the fourth coordinate of the color $c'(xy) = c'(uv) = c'$ guarantees that $c(x + y) = c(u + v)$. Thus, we have found a non-rainbow 3-AP in A, contradicting our assumptions. A symmetric argument shows that yu cannot be an edge of color c' either.

Let us count the number of edges of G. For every pair $(i, j), 1 \le i < j \le s$, there are at least $m^2 - m(2m)^\alpha > m^2 - 2m^{1+\alpha}$ edges between V_i and V_j. Indeed, for every $t \in [2m] \setminus A$, there are at most m pairs $(x,y) \in [m]^2$ such that $x+y=t$, and the number of such elements t is at most $(2m)^\alpha$. Hence, we have

$$|E(G)| \geq \binom{s}{2}(m^2 - 2m^{1+\alpha}) \geq \binom{n}{2} - n^{2-\gamma} - n^{1+\alpha-\alpha\gamma}.$$

The number of colors used by c' and, therefore, the number of induced matchings G can be partitioned into, is at most $s^2(2m)(2m)^\beta \leq 4n^{\beta+\gamma-\beta\gamma}$. This completes the proof of Corollary 2. □

4 Concluding remarks

Let us remark that in order to prove Corollary 2, it is enough to find a coloring of a large subset of $[n]$ such that in any 3-AP, the first and last elements have different colors. This can be achieved with slightly fewer colors: in the proof of Theorem 1, it is enough to define the coloring f as $f = (a_1, \dots, a_d)$ instead of $f = (a_1, \dots, a_d, b_1, \dots, b_d)$.

Our proof of Theorem 1 can be easily extended to longer arithmetic progressions.

Theorem 6. *For any positive integer k, there exist $\alpha, \beta < 1$ with the following property. For every sufficiently large positive integer n, there is a set $A \subset [n]$ with $|A| \geq n - n^\alpha$ and a coloring of A with at most n^β colors such that every arithmetic progression of length at most k in A is rainbow.*

Moreover, for every $\beta > 0$, we can choose $\alpha < 1$ satisfying the above conditions.

In order to establish Theorem 6, we need to modify the proof of Theorem 1 at the following two points.

1. We should construct an auxiliary coloring f on B such that if $f(\mathbf{x}) = f(\mathbf{y})$, then $\frac{p}{q}\mathbf{x} + (1 - \frac{p}{q})\mathbf{y} \in B$ for every $p, q \in [k]$. Color $(x_1, \dots, x_d)$ with the color $(a_1, \dots, a_d, b_1, \dots, b_d)$, where $a_i \in \{0, \dots, k! - 1\}$ such that $a_i \equiv x_i \mod k!$, and

$$b_i = \begin{cases} j & \text{if } \mathbf{x}(i) \leq \frac{C}{2} \text{ and } (\frac{k}{k-1})^{j-1} - 1 \leq \mathbf{x}(i) < (\frac{k}{k-1})^j - 1, \\ -j & \text{if } \mathbf{x}(i) > \frac{C}{2} \text{ and } (\frac{k}{k-1})^{j-1} - 1 \leq C - 1 - \mathbf{x}(i) < (\frac{k}{k-1})^j - 1. \end{cases}$$

   Then f uses $(k^k \log C)^{O(d)}$ colors.

2. Instead of Claim 4, we can show that if $\mathbf{x}_1, \dots, \mathbf{x}_k$ is a k-term arithmetic progression in S, then $|\mathbf{x}_1 - \mathbf{x}_k| \leq 10\sqrt{\epsilon} r$.

After these changes, the proof can be completed by straightforward calculations, in the same way as in the case of Theorem 1.

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