Cutting glass

Dedicated to Professor Branko Grünbaum on the occasion of his seventieth birthday

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Abstract

J. Urrutia asked the following question. Given a family of pairwise disjoint compact convex sets on a sheet of glass, is it true that one can always separate from one another a constant fraction of them using edge-to-edge straight-line cuts? We answer this question in the negative, and establish some lower and upper bounds for the number of separable sets. We also consider the special case when the family consists of intervals, axis-parallel rectangles, 'fat' sets, or 'fat' sets with bounded size.

1 Introduction

Let P be a subset of the plane, and let H_1 and H_2 be the two open half-planes bounded by a straight line ℓ. *Cutting* P along ℓ, we obtain two pieces $P_1 = P \cap H_1$ and $P_2 = P \cap H_2$. We say that m pairwise disjoint sets in the plane are *separable* if we can cut the plane into two parts, and successively cut each part into smaller pieces until we obtain m pieces, each containing precisely one of our m sets. (See Figure 1.)

For two positive functions defined on the positive integers, we use the notation $f(n) = \Omega(g(n))$ to express that $f(n) > cg(n)$ for some positive constant c.

Jorge Urrutia [U96] raised the following problem. Is it true that any family of n pairwise disjoint compact convex sets in the plane has at least $\Omega(n)$ separable members?

In the following special case, the answer is easily seen to be in the affirmative.

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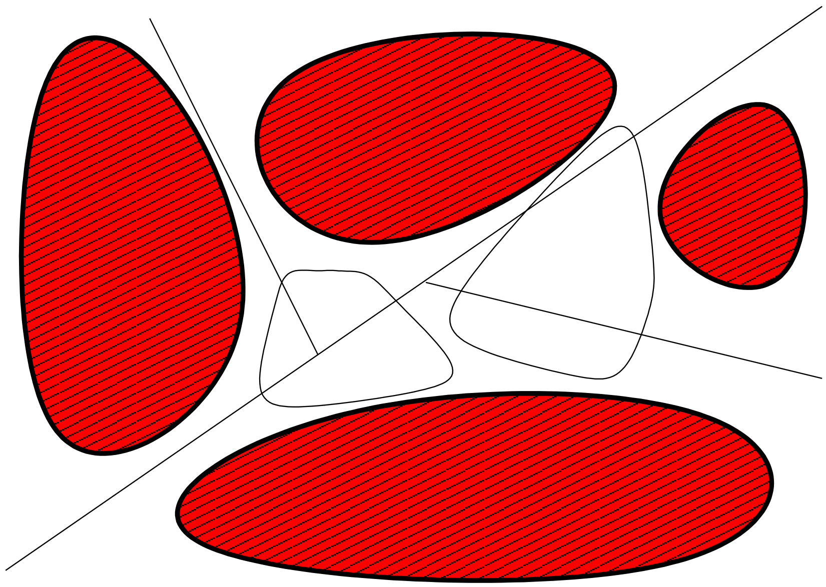


Figure 1.

Proposition 1.1 *Let $R > r > 0$ be fixed, and let $\mathcal{F}$ be a family of n pairwise disjoint compact convex sets in the plane, each containing a circle of radius r and contained in another circle of radius R.*

Then $\mathcal{F}$ has at least cn separable members, where $c = c(r, R) > 0$.

Proof: Choose j at random, uniformly in $[0, 8R]$, and cut the plane into squares along the lines $x = 8Ri + j$ and $y = 8Ri + j$ for all integers i. The expected number of members of $\mathcal{F}$ intersected by these lines is at most $n/2$. Since there are at most $64R^2/(r^2\pi)$ members of $\mathcal{F}$ contained in the same square, we can find a separable subfamily of size at least $(r^2\pi/(128R^2))\, n$. □

The above statement does not remain true without the assumption on the circumradii and inradii of the members of $\mathcal{F}$. That is, the answer to Urrutia's question, in full generality, is in the negative.

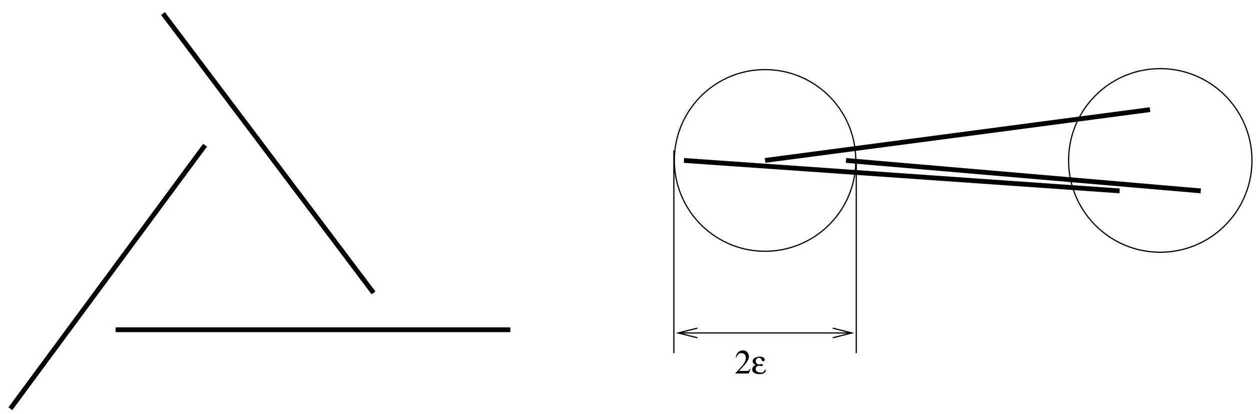
Theorem 1.2 *There exists a family of n pairwise disjoint straight-line segments in the plane such that all separable subfamilies are of size $O(n^{\log 2/\log 3})$.*

Proof: It is enough to show that for every positive integer k there is a family C_k of 3^k disjoint intervals, such that C_k has at most 2^k separable members.

We construct C_k recursively. During the construction we make sure that the endpoints of the intervals in C_k are in general position: no three of them lie on the same straight line.

The case $k = 1$ is shown on Figure 2(a). Notice that, for an arbitrary segment pq and any $\varepsilon > 0$, the construction can be carried out in such a way that each of the three segments has one of its endpoints in the disk of radius ε around p and the other in the disk of radius ε around q (see Figure 2(b)). Denote such a configuration $C_1(p, q, \varepsilon)$.

For $k > 1$, replace each segment pq of C_{k-1} by $C_1(p, q, \varepsilon)$. If $\varepsilon > 0$ is sufficiently small, the resulting family C_k consists of disjoint segments. Consider a separable subfamily $\mathcal{F}$. It contains at most two members from each of the sets $C_1(p, q, \varepsilon)$. Let $\mathcal{F}'$ be the members pq of C_{k-1} such that $\mathcal{F}$ contains at least one element from $C_1(p, q, \varepsilon)$. Recall that the segments of C_{k-1} are in general position. Thus, if ε is sufficiently small, then $\mathcal{F}'$ is also separable. Thus, $|\mathcal{F}| \leq 2|\mathcal{F}'| \leq 2^k$. □



(a) (b)

Figure 2.

Next we state some positive results.

Theorem 1.3 *Any family of n pairwise disjoint compact convex sets in the plane has at least $\Omega(n^{1/3})$ separable members.*

The construction proving Theorem 1.2 uses only segments. For families of segments, the estimate in Theorem 1.3 can be improved.

Theorem 1.4 *Any family of n pairwise disjoint straight-line segments in the plane has at least $\Omega(n^{1/2})$ separable members.*

It seems plausible that, for families of axis-parallel rectangles, the answer to Urrutia's question is in the affirmative. We can only prove a somewhat weaker result.

Theorem 1.5 *Any family of n pairwise disjoint axis-parallel rectangles in the plane has* $\Omega(n/\log n)$ *separable members.*

We believe that among the worst possible families of n convex sets from the point of view of separability, i.e., among those which have the fewest number of separable members, there is one which contains only straight-line segments. This conjecture is supported by the fact that for families of not too 'longish' sets, we can establish much stronger results than Theorem 1.4.

A family $\mathcal{F}$ of plane convex sets is called ε*-fat* if, for each member of $\mathcal{F}$, the ratio of the inradius and the circumradius is at least ε (cf. [MMP91]).

Theorem 1.6 *For any* $\varepsilon > 0$*, there exists a constant* $c_\varepsilon > 0$ *such that every* ε*-fat family of* n *pairwise disjoint compact convex sets in the plane has at least* $c_\varepsilon n/\log n$ *separable members.*

The proofs of Theorems 1.3–1.5 are presented in Section 2. Section 3 contains the proof of the last theorem and a corollary. In Section 4, we improve Theorem 1.6 in the special case when the ratio of the sizes of the largest vs. the smallest members of a family of size n is bounded, say, by a polynomial of n. Unfortunately, the improved bound given in Theorem 4.1 is still sublinear in n, unless the ratio is bounded by a constant, in which case Theorem 4.1 reduces to Proposition 1.1. In Section 5, we discuss analogous questions in higher dimensions, while the last section contains a few concluding remarks.

2 Proofs of Theorems 1.3–1.5

In order to establish Theorem 1.3, we need four simple but useful observations. As usual, we fix an orthogonal system of coordinates (x, y) in the plane, and call the directions of the x-axis and the y-axis *horizontal* and *vertical*, respectively.

Lemma 2.1 *Given* n *compact convex sets in the plane and a positive integer* $k < n$*, there exists a vertical line* ℓ *satisfying at least one of the following two conditions:*
(i) ℓ *intersects at least* $k+1$ *sets;*
(ii) both half-planes bounded by ℓ *contain at least* $(n-k)/2$ *sets.*

Proof: For a compact set F in the family consider the largest x coordinate x_F of a point in the set and let x_0 be the $\lceil (n-k)/2 \rceil$-th smallest of these values. Either (i) holds for the line $x = x_0$ or the number of sets lying on the right-hand side of $x = x_0 + \varepsilon$ for a small enough $\varepsilon > 0$ is at least $n - \lceil (n-k)/2 \rceil - k + 1 \geq \lceil (n-k)/2 \rceil$, in which case (ii) is true. □

A set of intervals on the line is said to be *nested*, if any pair of its elements are comparable by inclusion. In particular, the intervals of a finite nested set have a point in common.

Lemma 2.2 *Let $\mathcal{F}$ be a family of k pairwise disjoint compact convex sets in the plane, whose orthogonal projections onto the x-axis form a nested set of intervals.*

Then $\mathcal{F}$ has at least $(k+3)/4$ separable members.

Proof: The proof is by induction on k. For $k \leq 2$, the assertion is obviously true. Let $k \geq 3$, and assume that we have already established the statement for all integers smaller than k. Let F_i denote the member of $\mathcal{F}$ with the i-th largest projection onto the x-axis, and let p_i and q_i be (one of) the leftmost and rightmost points of F_i, respectively ($1 \leq i \leq k$). Clearly, the line p_iq_i does not intersect any F_j with $j > i$.

Assume first that, for some $i \leq 3$, both half-planes bounded by p_iq_i fully contain at least one member of $\mathcal{F}$; say, one of these half-planes contains $k_1 \geq 1$, the other $k_2 \geq 1$ members, where $k_1 + k_2 \geq k - 3$. Applying the induction hypothesis to these k_1 and k_2 members, respectively, we obtain that $\mathcal{F}$ has at least

$$\frac{k_1+3}{4} + \frac{k_2+3}{4} = \frac{k_1+k_2+6}{4} \geq \frac{k+3}{4}$$

separable members, as required.

Thus, we can suppose that, for every $i \leq 3$, at least one of the half-planes bounded by p_iq_i does not contain any member of $\mathcal{F}$. In this case any one of F_1, F_2 and F_3 must be below all F_j for $j > 3$ or it must be above all F_j for $j > 3$ in the ordering of $\mathcal{F}$ according to the y-coordinates of the intersections of its members with a vertical line passing through all of them. In this ordering either the highest two or the lowest two positions are occupied by members of $\{F_1, F_2, F_3\}$. Any line separating this consecutive pair may intersect the third one but must avoid every other set F_j. Using the induction hypothesis again, we can conclude that $\mathcal{F}$ has at least $1 + \frac{(k-2)+3}{4} > \frac{k+3}{4}$ separable members. □

Lemma 2.3 *Let $(F_1, \ldots, F_k)$ be a sequence of pairwise disjoint compact convex sets in the plane, intersecting a vertical line in this order. Let p_iq_i denote the orthogonal projection of F_i onto the x-axis, and assume that $p_1 < p_2 < \ldots < p_k < q_1 < q_2 < \ldots < q_k$.*

Then $F_1, \ldots, F_k$ are separable.

Proof: According to our notation, for any $i < j$, F_i lies 'below' F_j. For each i, $1 \leq i < k$, pick a line ℓ_i that separates F_i from F_{i+1}. It is easy to check that a line ℓ_j with *minimum slope* cannot intersect any F_i. Thus, ℓ_j separates $\{F_1, \ldots, F_j\}$ from $\{F_{j+1}, \ldots, F_k\}$, and these two subfamilies are separable, recursively. □

The following well known statement can be regarded as a special case to Dilworth's theorem [D50] on partially ordered sets.

Lemma 2.4 [ES35] *Let k_1 and k_2 be positive integers. Any sequence of $k_1k_2 + 1$ reals contains a monotone increasing subsequence of length $k_1 + 1$ or a monotone decreasing subsequence of length $k_2 + 1$.* □

Now we are in a position to complete the

Proof of Theorem 1.3: Let $m(n)$ denote the maximum number m such that every family of n pairwise disjoint compact convex sets in the plane has m separable members. We prove by induction that $m(n) \geq n^{1/3}/2$.

The base case $n = 2$ is trivial. Suppose that $n > 2$ and that we have already proved the claim for all positive integers smaller than n. Fix any family $\mathcal{F}$ of n pairwise disjoint compact convex sets in the plane.

Assume first that there is a vertical line ℓ intersecting at least $k := \lceil n/2 \rceil$ members $F_1, \ldots, F_k \in \mathcal{F}$. Let $p_i q_i$ be the orthogonal projection of F_i ($p_i < q < i$, $1 \leq i \leq k$). Renumbering the sets, if necessary, we can assume that $p_1 < p_2 < \cdots < p_k$. According to Lemma 2.4,
(a) there is a strictly increasing sequence i_j, $j = 1, 2, \ldots, k_1 := \lceil n^{2/3}/4 \rceil$, such that q_{i_j} is monotone increasing; or
(b) there is a strictly increasing sequence i_j, $j = 1, 2, \ldots, k_2 := \lceil 2n^{1/3} \rceil$, such that q_{i_j} is monotone decreasing.

In case (a), applying Lemma 2.4 again, we obtain that (i_j) has a subsequence $i_{j(1)} < i_{j(2)} < \ldots$ of length $\lceil n^{1/3}/2 \rceil$ with the property that ℓ meets $F_{i_{j(1)}}, F_{i_{j(2)}}, \ldots$ in this (or in the opposite) order. In view of Lemma 2.3, these sets are separable.

In case (b), the orthogonal projections of F_{i_j} onto the x-axis, $j = 1, 2, \ldots, k_2$, form a nested family of intervals. Lemma 2.2 implies that there are at least $(k_2 + 3)/4 \geq n^{1/3}/2$ separable members.

Thus, we can assume that no vertical line intersects $k = \lceil n/2 \rceil$ members of $\mathcal{F}$. In this case, by Lemma 2.1, there are two subfamilies $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, each of size at least $n/4$, which can be separated by a vertical line. Applying the induction hypothesis to $\mathcal{F}_1$ and $\mathcal{F}_2$, we obtain that $\mathcal{F}$ has at least

$$2m\left(\lceil n/4 \rceil\right) \geq 2(n/4)^{1/3}/2 > n^{1/3}/2$$

separable members, as required. □

Proof of Theorem 1.4: Let $\mathcal{F}$ be a family of n pairwise disjoint closed segments in the plane. We prove by induction on n that $\mathcal{F}$ has at least $\sqrt{n/2}$ separable members.

For $n = 1, 2$, there is nothing to prove. So we can suppose that $n > 2$ and that we have already proved the claim for all families having fewer than n members. If there is no vertical line intersecting at least $k := \lceil n/2 \rceil$ members of $\mathcal{F}$, then $\mathcal{F}$ has two $\lceil n/4 \rceil$-membered subfamilies separated by a line, and the proof can be completed in exactly the same way as that of Theorem 1.3.

Thus, we can assume that there is a vertical line ℓ intersecting at least k members, $F_1, \ldots, F_k \in \mathcal{F}$, numbered from bottom to top in the order of their intersections with ℓ. By Lemma 2.4, there is a strictly increasing sequence i_j, $j = 1, 2, \ldots, h := \lceil \sqrt{k} \rceil$, such that the slopes of the lines containing $F_{i_1}, \ldots, F_{i_h}$ form a monotone increasing or monotone decreasing sequence. If this sequence is monotone increasing (decreasing), consider the segment F_{i_j} extending farthest to the left (right), and notice that the line containing it cannot meet any other member of $\mathcal{G} = \{F_{i_1}, \ldots, F_{i_h}\}$. Therefore, a line running parallel and very close to F_{i_j} will still be disjoint from all members of $\mathcal{G}$ and will separate them into two non-empty groups. Recursively, both of these groups are separable, and so is $\mathcal{G}$. Thus, $\mathcal{F}$ has $|\mathcal{G}| = h = \lceil\sqrt{k}\rceil \geq \sqrt{n/2}$ separable members. □

Proof of Theorem 1.5: Let $\overline{m}(n)$ denote the largest number $\overline{m}$ such that any family of n pairwise disjoint axis-parallel rectangles in the plane has $\overline{m}$ separable members. In view of the fact that any family of pairwise disjoint axis-parallel rectangles intersecting the same vertical line is separable by horizontal cuts, Lemma 2.1 yields the recurrence relation

$$\overline{m}(n) \geq \max_{0<k<n} \min\left(k, 2\overline{m}\left(\left\lceil\frac{n-k}{2}\right\rceil\right)\right).$$

This immediately implies that $\overline{m}(n) \geq n/(2\log_2 n)$. □

3 Proof of Theorem 1.6

We say that a family of sets permits a *line transversal*, if all of its members can be intersected by a line. The proof of Theorem 1.5 works for any family of sets, $\mathcal{F}$, satisfying the condition that every subfamily $\mathcal{G} \subseteq \mathcal{F}$ with a vertical line transversal has at least $c|\mathcal{G}|$ separable members (where $c > 0$ is a constant). Therefore, to prove Theorem 1.6, it is sufficient to establish the following.

Theorem 3.1 *For any $\varepsilon > 0$, there exists a constant $d = d(\varepsilon) > 0$ such that every family of n pairwise disjoint convex compact ε-fat sets in the plane, which permits a line transversal, has at least dn separable members.*

Proof: Let $0 < \varepsilon < 1/10$ be fixed, and let $\mathcal{F}$ be a family of n pairwise disjoint compact convex ε-fat sets in the plane, all of which intersect the y-axis, say.

For any $F \in \mathcal{F}$, let $r(F)$ and $R(F)$ denote the inradius and circumradius of F, respectively. By the assumption, $r(F)/R(F) \geq \varepsilon$. The intersection of F with the y-axis is a segment $a_F b_F$ whose lower endpoint is a_F and upper endpoint is b_F. Choose two tangent lines to F at a_F and b_F, and denote the (smallest) counter-clockwise angles from the y-axis to these lines by α_F and β_F, respectively ($\alpha_F, \beta_F \in (0, \pi)$). In case all of F has non-negative (respectively non-positive) x-coordinates we set $\alpha_F = 0$, $\beta_F = \pi$ (respectively $\alpha_F = \pi$, $\beta_F = 0$. Notice that, if $\alpha_F = \beta_F$, then F must lie in a parallel strip whose vertical cross-section is of length $b_F - a_F$, and so $2r(F) \leq (b_F - a_F)\sin\alpha_F$. In general, we have

$$r(F) \leq \frac{b_F - a_F}{2}\sin\alpha_F + R(F)\sin|\beta_F - \alpha_F| \tag{1}$$

Partition the elements $F \in \mathcal{F}$ into a constant number (at most $\lceil 100\pi/\varepsilon \rceil^2$) of classes, according to the values $\lfloor 100\alpha_F/\varepsilon \rfloor$ and $\lfloor 100\beta_F/\varepsilon \rfloor$. Let $\mathcal{F}_0$ be one of the largest classes, i.e., $|\mathcal{F}_0| = \Omega(n)$. We distinguish two cases.

CASE A: There is an interval $I \subseteq [0, \pi)$ of length $\varepsilon/10$ such that, for every $F \in \mathcal{F}_0$, we have $\alpha_F, \beta_F \in I$.

CASE B: There are two intervals, I_1 and I_2, each of length $\varepsilon/100$, which are at least $9\varepsilon/100$ apart, and $\alpha_F \in I_1$ and $\beta_F \in I_2$ for every $F \in \mathcal{F}_0$.

It is sufficient to prove that, if $|\mathcal{F}_0| \geq 2$, then in both cases we can find a *separating line* (i.e., a straight line having at least one member of $\mathcal{F}_0$ on both of its sides) which meets at most five members of $\mathcal{F}_0$. Indeed, cutting along such a line ℓ_0, and recursing on the subfamilies lying in the two complementary half-planes bounded by ℓ_0, we obtain that $\mathcal{F}_0$ has at least $(|\mathcal{F}_0|+5)/6$ separable members, which will complete the proof of the lemma. (We applied exactly the same argument in the proof of Lemma 2.2.) Notice that the existence of the separating line is trivial for $2 \leq n \leq 7$ so we may suppose $n \geq 8$.

In CASE A, (1) implies that, for every $F \in \mathcal{F}_0$,

$$\varepsilon R(F) \leq r(F) \leq \frac{b_F - a_F}{2} \sin \alpha_F + R(F) \sin \frac{\varepsilon}{10}. \tag{2}$$

Using the fact that $\sin \frac{\varepsilon}{10} < \frac{\varepsilon}{10}$, we have

$$R(F) \leq \frac{5}{9\varepsilon} (b_F - a_F) \sin \alpha_F.$$

Plugging the relation $b_F - a_F \leq 2R(F)$ into (2), we also obtain that

$$\sin \alpha_F \geq \frac{9}{10}\varepsilon,$$

which shows that in CASE A the interval I cannot be closer to 0 or to π than $(9/10)\varepsilon - (1/10)\varepsilon = (4/5)\varepsilon$.

Fix a line ℓ that can be reached from the y-axis by a counter-clockwise turn through an angle belonging to I, and project every member of $\mathcal{F}$ to the y-axis parallel to ℓ. Let the projection of F be $a'_F b'_F$, where $a'_F \leq b'_F$. Obviously, $a_F b_F \subseteq a'_F b'_F$. It follows from the last two inequalities, using the law of sines, that

$$\max(a_F - a'_F, b'_F - b_F) < \frac{b_F - a_F}{10}. \tag{3}$$

Let $\mathcal{F}_1$ be the family consisting of those three members $F \in \mathcal{F}_0$, whose intersections with the y-axis, $a_F b_F$, are the longest (break ties arbitrarily). (3) implies that no member of $\mathcal{F}_0 \setminus \mathcal{F}_1$ can intersect any of the three straight lines parallel to ℓ, passing through the midpoints of the segments

$a_F b_F$, $F \in \mathcal{F}_1$. If one of these three lines is a separating line, we are done. Otherwise, there are two possibilities:
(i) two members of $\mathcal{F}_1$ occupy the two highest positions, or
(ii) two members of $\mathcal{F}_1$ occupy the two lowest positions in the ordering of the members of $\mathcal{F}_0$ according to the y-coordinates of their intersections with the y-axis.

Suppose without loss of generality that (i) holds, and let F_1 and F_2 denote the members of $\mathcal{F}_1$ occupying the highest and the second highest positions, respectively. One can find a straight line ℓ_0 in a direction in I separating F_1 from F_2. To see this blow up the the two sets, each from one of its points, until they touch each other. One can find ℓ_0 through the intersection of the enlarged sets. Using (3) for the projections in the direction of ℓ_0, one can verify again that ℓ_0 cannot intersect any member of $\mathcal{F}_0$ except perhaps the third member of $\mathcal{F}_1$. This completes the proof in CASE A.

In CASE B, suppose without loss of generality that $I_1 = [\alpha', \alpha]$, $I_2 = [\beta, \beta']$, where $0 \le \alpha' < \alpha < \beta < \beta' \le \pi$. Let $I := [\alpha, \beta]$. Note that in this case, for any $F \in \mathcal{F}_0$, the tangents to F at a_F and b_F must intersect in the left half-plane $x \le 0$. It is easy to see that, if the direction of a line ℓ is in I, then in the left half-plane ℓ can intersect at most one member of $\mathcal{F}_0$.

For any $F \in \mathcal{F}_0$, let p_F be a rightmost point of F. The distance of p_F from the y-axis is called the *depth* of F. Assign a line ℓ_F to F, as follows.

1. Let ℓ_F be any line through p_F, whose angle with the y-axis belongs to I and which intersects the segment $a_F b_F$, if such a line exists.
2. If no such line exists, then either the line ℓ_α through a_F in direction α passes above p_F or the line ℓ_β through b_F in direction β passes below p_F. Set $\ell_F := \ell_\alpha$ or $\ell_F := \ell_\beta$, respectively.

Since ℓ_F intersects F in the left half-plane $x \le 0$, it cannot intersect any other member of $\mathcal{F}_0$ in the left half-plane. If ℓ_F intersects some other member $G \ne F$ of $\mathcal{F}_0$ in the right half-plane, then the depth of G must be larger than the depth of F. If ℓ_F passes through p_F, then this is obvious, otherwise, it follows from the fact that F is ε-fat.

Let $\mathcal{F}_1$ be the family consisting of those five members of $\mathcal{F}_0$, whose depths are the largest (break ties arbitrarily). By the above observation, the lines ℓ_F for $F \in \mathcal{F}_1$ cannot intersect any member of $\mathcal{F}_0 \setminus \mathcal{F}_1$. Thus, if any of them is a separating line (i.e., has at least one member of $\mathcal{F}_0$ on both of its sides), then we are done. Otherwise, we can finish the proof similarly as in CASE A. That is, we can assume that
(i) three members of $\mathcal{F}_1$ occupy the three highest positions, or
(ii) three members of $\mathcal{F}_1$ occupy the three lowest positions in the ordering of the members of $\mathcal{F}_0$ according to the y-coordinates of their intersections with the y-axis.

Assume with no loss of generality that (i) holds, and denote the three members at the highest positions by F_1, F_2, and F_3, in this order. We may also suppose that the depth of F_2 is greater than that of F_3 as otherwise ℓ_{F_3} would be separating. From the fact that F_2 is ε-fat it follows that there is a straight line separating F_1 from F_3, whose direction is in the interval $(0, \beta]$, and that any such line is disjoint from all members of $\mathcal{F}_0 \setminus \mathcal{F}_1$.

This settles CASE B and finishes the proof of Theorem 3.1 and hence Theorem 1.6. □

The following result is a direct corollary of Theorem 1.6.

Theorem 3.2 *Any family of n pairwise disjoint homothetic copies of a compact convex set F in the plane has at least $cn/\log n$ separable members, where c is a positive constant not depending on F.*

Proof: If F is a segment, the entire family is separable. Otherwise, there is an affine transformation of the plane which takes F into a convex body, whose circumradius is at most twice larger than its inradius (consider the Löwner-Johns ellipse [G63]). The proof now follows from the observation that the separability problem is invariant under affine transformations. □

4 Separation of fat sets with bounded size

As we noted in the Introduction, it seems plausible that any family of n pairwise disjoint axis-parallel rectangles in the plane has $\Omega(n)$ separable members. However, we were unable to verify this even for axis-parallel squares. We include the following modest improvement on Theorem 1.6 in case the sizes of the sets do not vary too much.

In order to achieve this improvement, we need to bound the *variance* of the sizes of our sets, i.e., to put an upper bound V on the ratio of the circumradii of the largest and smallest members of the family.

Theorem 4.1 *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ with the following property.*

Any family $\mathcal{F}$ of n pairwise disjoint compact convex ε-fat sets in the plane contains at least $C_\varepsilon n \log\log V/\log V$ separable members, where $V > 2$ is an upper bound of the ratio of the circumradii of any two sets in $\mathcal{F}$.

The bound given in the above theorem is sublinear, unless the variance V of the family is bounded from above by some constant. For constant V, Theorem 4.1 reduces to Proposition 1.1. If V grows polynomially in n, Theorem 4.1 gives a slightly better bound than Theorem 1.6. However, for large variance V, Theorem 1.6 is stronger, as its statement is independent of V.

The somewhat weaker bound $C_\varepsilon n/\log V$ can be easily deduced from Proposition 1.1. Indeed, notice that scaling shows that the constant $c = c(r, R)$ in Proposition 1.1 depends only on the ratio r/R. If $\mathcal{F}$ satisfies the conditions in Theorem 4.1, it can be partitioned into $\lceil \log V \rceil$ 'uniform' subfamilies such that within each subfamily the variance is at most 2, therefore the circumradius of any member is at most $2/\varepsilon$ times the inradius of any other member. Applying Proposition 1.1 to the largest uniform subfamily, the weaker bound follows. (Throughout this section, all logarithms will be base 2.)

The idea of the proof is that we first cut the plane into appropriate size squares so that many members of $\mathcal{F}$ are fully contained in one of these cells, but not too many lie in the same cell. Then we apply Theorem 1.6 within each cell, separately.

Proof: Let us denote the circumradius of a set F by $R(F)$. Without loss of generality we may assume that $V = 2^k$, for a positive integer k, and $1 \leq R(F) < 2^k$, for any set $F \in \mathcal{F}$. We partition $\mathcal{F}$ into k subfamilies, as follows. For every i, $1 \leq i \leq k$, let

$$\mathcal{F}_i = \{F \in \mathcal{F} \mid 2^{i-1} \leq R(F) < 2^i\}.$$

Claim A. *There exists an integer $1 \leq a \leq k$ such that*
(i) $\sum_{i=1}^{a} |\mathcal{F}_i| \geq n/2$,
(ii) $\sum_{i=b+1}^{a} |\mathcal{F}_i| \geq \frac{a-b}{2k} n$ *for every* $1 \leq b < a$.

Define recursively a sequence $a_0 > a_1 > a_2 > \ldots$, as follows. Set $a_0 := k$. If a_{j-1} has already been defined for some j, choose a_j to be a non-negative integer smaller than a_{j-1} such that

$$\sum_{i=a_j+1}^{a_{j-1}} |\mathcal{F}_i| < \frac{a_{j-1} - a_j}{2k} n.$$

If there is no such integer, we stop. Let a_t be the last element of this sequence.

Clearly, $a := a_t$ satisfies (ii). As for condition (i), notice that

$$\sum_{i=1}^{a} |\mathcal{F}_i| = |\mathcal{F}| - \sum_{j=1}^{t} \sum_{i=a_j+1}^{a_{j-1}} |\mathcal{F}_i| \geq n - \sum_{j=1}^{t} \frac{a_{j-1} - a_j}{2k} n = n - \frac{k-a}{2k} n \geq n/2.$$

This inequality proves Claim A.

Fix an integer a satisfying the conditions of Claim A. Let $\mathcal{F}' = \cup_{i=1}^{a} \mathcal{F}_i$. Assign weights to the elements of $\mathcal{F}'$ in the following way. For $i = 1, \ldots, a$, assign the weight

$$w(F) := \frac{1}{a+1-i}$$

to every member $F \in \mathcal{F}_i$.

Claim B. *The total weight assigned to the elements of $\mathcal{F}'$ is at least $\frac{n \log k}{4k}$, provided that n is sufficiently large.*

Let W denote the total weight of the elements in $\mathcal{F}'$. By Claim A, we have

$$W = \sum_{i=1}^{a} \frac{|\mathcal{F}_i|}{a+1-i} = \frac{\sum_{i=1}^{a} |\mathcal{F}_i|}{a} + \sum_{b=1}^{a-1} \left(\sum_{i=b+1}^{a} |F_i| \left(\frac{1}{a-b} - \frac{1}{a+1-b} \right) \right)$$

$$\geq \frac{n/2}{a} + \sum_{b=1}^{a-1} \left(\frac{a-b}{2k} n \left(\frac{1}{a-b} - \frac{1}{a+1-b} \right) \right) = \frac{n}{2a} + \frac{n}{2k} \sum_{b=1}^{a-1} \frac{1}{a+1-b} > \frac{n}{2a} + \frac{n(\log a - 1)}{3k}.$$

For small values of a, the first term of the last expression exceeds the bound stated for W, for large values of A, the second term does. This proves Claim B.

We proceed similarly as in the proof of Proposition 1.1. Cut the plane along all horizontal and vertical lines $x = i2^{a+3} + j$ and $y = i2^{a+3} + j$, where i runs over the integers and j is selected at random, uniformly from the interval $[0, 2^{a+3}]$. The probability that a given member of $\mathcal{F}'$ is not met by any of these lines is at least $1/2$, since the circumradius of the sets in $\mathcal{F}'$ is bounded by 2^a. Hence, by the linearity of the expectation, the expected total weight of the *intact* (i.e., uncut) members of $\mathcal{F}'$ is at least half of the total weight of *all* members. There is a particular choice of j, for which the total weight of the family $\mathcal{G}$ of all intact members of $\mathcal{F}'$ is at least as large as its expectation. According to Claim B, we have

$$\sum_{F \in \mathcal{G}} w(F) \geq \frac{n \log k}{8k}. \tag{4}$$

After this first round of cuts, the plane falls into squares of side-length 2^{a+3}. Fix one such piece, and denote by $\mathcal{G}_0$ the family of all members of $\mathcal{G}$ belonging to it. Let $W_0 := \sum_{F \in \mathcal{G}_0} w(F)$ be the total weight of the elements in $\mathcal{G}_0$.

Recall that all members of $\mathcal{F}$ (and thus of $\mathcal{G}_0$) are ε-fat with some constant $\varepsilon > 0$, and that $k = \log V$, where V was the upper bound for the 'variance' of the set sizes in $\mathcal{F}$.

We can now finish the proof of Theorem 4.1 by combining (4) with the following

Claim C. *$\mathcal{G}_0$ has a separable subfamily whose size is at least $\bar{c}W_0 = \bar{c}\sum_{F \in \mathcal{G}_0} w(F)$, for some constant $\bar{c} = \bar{c}_\varepsilon > 0$ depending only on ε.*

To verify the claim, let $m := |\mathcal{G}_0|$ and notice that, if $m > 1$, then Theorem 1.6 guarantees the existence of a separable subfamily of size $c_\varepsilon(m/\log m)$ in $\mathcal{G}_0$. It remains to show that $\frac{m}{\log m} \geq cW_0$, for a suitable constant $c > 0$ depending on ε.

Since $\mathcal{G}_0$ consists of pairwise disjoint sets packed into a square of side-length 2^{a+3}, and the area of an ε-fat set F is at least $(\varepsilon R(F))^2\pi \geq \varepsilon^2\pi 4^{a-2/w(F)}$, we have

$$4^{a+3} \geq \varepsilon^2 \pi \sum_{F \in \mathcal{G}_0} 4^{a-2/w(F)}.$$

On the other hand, using the convexity of the function $4^{a-2/w}$ over the interval $w = w(F) \in (0,1]$, we obtain that the right-hand side of the above inequality is at least $\varepsilon^2\pi m 4^{a-2m/W_0}$. Thus, we have

$$4^{a+3} \geq \varepsilon^2 \pi m 4^{a-2m/W_0}.$$

Taking logarithms, it follows that for large enough m

$$W_0 \leq \frac{4m}{\log m + 2\log\varepsilon + \log\pi - 6} = O(m/\log m),$$

as required. □

5 Higher dimensions

The definition of a *separable family* can be naturally extended to higher dimensions $d > 2$. We say that m pairwise disjoint sets in d-space are separable, if we can cut the space by a hyperplane into two parts, and successively cut each part into smaller pieces until we obtain m pieces, each containing precisely one of our m sets.

In general, it is not true even in 3-space that every infinite family of pairwise disjoint convex sets has three separable members. Indeed, as noted in [T79], given a family of infinitely many disjoint straight lines in 3-space, no three of which are parallel to the same plane, any plane separating two members of the family must cross the remaining lines. To obtain a family of *compact* convex sets with this property, one can clip each member in a finite subfamily of the above construction by a ball around the origin, whose radius is sufficiently large.

However, for fat sets and axis-parallel boxes, it is not hard to establish some positive results.

Theorem 5.1 *Any family of n pairwise disjoint compact convex ϵ-fat sets in d-space has a separable subfamily of at least $cn/(\log n)^d$ members, where $c = c(\epsilon, d) > 0$ is a constant depending only on ϵ and d.*

The proof is based on the following

Lemma 5.2 *Let $\mathcal{F}$ be a family of n pairwise disjoint compact convex ϵ-fat sets in d-space such that each of them intersects all the d coordinate hyperplanes.*

Then $\mathcal{F}$ has a separable subfamily of at least $c'n$ members, where $c' = c'(\epsilon, d) > 0$ is a constant depending only on ϵ and d.

Proof: First, note that, if a set F in d-space intersects all coordinate hyperplanes and it has a point at distance r from the origin O, then the diameter of F is at least $r/\sqrt{d}$. Next, notice that, if F is an ϵ-fat convex set of diameter $d > s$, and x is a point of F, then $F \cap B(x, s)$, the intersection of F with the ball of radius s centered at x, contains a ball of radius $\epsilon s/2$. Indeed, we obtain such a ball by shrinking the inscribed ball of F from x to a fraction s/d of its original size.

Hence, any member of $\mathcal{F}$, which has a point at distance r from the origin, contains a ball of radius $\epsilon r/(2\sqrt{d})$, lying entirely within $B(O, 2r)$. As the sets $F \in \mathcal{F}$ are pairwise disjoint, no more than $(4\sqrt{d})^d$ of these balls fit into $B(O, 2r)$. Consequently, $\mathcal{F}$ has at most $(4\sqrt{d})^d$ members that have at least one point at distance r from the origin. This immediately implies the existence of a subfamily $\mathcal{F}' = \{F_1, \ldots, F_m\} \subset \mathcal{F}$ with $m \geq n/(4\sqrt{d})^d$, such that every point of F_i is closer to the origin than any point of F_{i+1} $(i = 1, \ldots, m-1)$. To see that $\mathcal{F}'$ is *separable*, it is enough to observe that, if the largest ball B_i around the origin that does not overlap F_i touches F_i at a point p_i, then the tangent hyperplane to B_i at the point p_i separates F_i from every F_j, $j < i$. □

Proof of Theorem 5.1: We establish the stronger claim that, for any $\epsilon > 0$ and for any $d \geq i \geq 0$, there is a constant $c'' = c''(\epsilon, d, i) > 0$ such that every family of n pairwise disjoint compact convex ϵ-fat sets in the plane, all of whose members intersect the first i coordinate hyperplanes, has a separable subfamily with at least $c''n/(\log n)^{d-i}$ members.

The proof is by induction on $d - i$. The base case, $i = d$, was settled in Lemma 5.2. The case $i = 0$ gives the theorem.

Let $f(n, \epsilon, d, i)$ be the minimum size of the largest separable subfamily in a family of n pairwise disjoint compact convex ϵ-fat sets in d-space, all of which meet the first i coordinate hyperplanes. Assume we have already verified the claim for some i $(d \geq i > 0)$, and next we wish to prove it for $i - 1$. Let $\mathcal{F}$ be a family of n pairwise disjoint compact convex ϵ-fat sets in d-space, all meeting the first $i-1$ coordinate hyperplanes, and assume that $\mathcal{F}$ ha only $f(n, \epsilon, d, i-1)$ separable members. As in Lemma 2.1, for every $1 \leq j \leq n/2$, one can find a hyperplane $x_i = z$ such that either it intersects at least $n - 2j + 2$ members of $\mathcal{F}$, or both half-spaces bounded by it contain at least j members of $\mathcal{F}$. In the former case, we can translate this hyperplane to the ith coordinate hyperplane (not affecting the number of separable members) and obtain that $f(n, \epsilon, d, i-1) \geq f(n - 2j + 2, \epsilon, d, i)$. In the latter case, first cutting along the hyperplane $x_i = z$ and then dealing separately with the families on either side of it, we obtain $f(n, \epsilon, d, i-1) \geq 2f(j, \epsilon, d, i-1)$. Thus, we have

$$f(n, \epsilon, d, i-1) \geq \min\left(f(n - 2j + 2, \epsilon, d, i), 2f(j, \epsilon, d, i-1)\right).$$

To finish the proof of the claim, set $j = \lfloor n/2 - n/\log n \rfloor$ and use the induction hypothesis on $f(n - 2j + 2, \epsilon, d, i)$. □

Theorem 5.3 *Any family of n pairwise disjoint axis-parallel boxes in d-space has a separable subfamily with at least $c'''n/(\log n)^{d-1}$ members, where $c''' = c'''(\epsilon, d) > 0$ is a constant depending only on ϵ and d.*

Proof: The proof can be carried out along the lines of the last argument. Alternatively, one can also prove Theorem 5.3 by induction on d, as separating a family of d-dimensional boxes intersecting a coordinate hyperplane reduces to a similar $d-1$-dimensional problem. □

6 Remarks

6.1 It is a natural first approach to our problem to try to find a line cutting through relatively few members of the family $\mathcal{F}$ and separating the others into two large subfamilies. Then, recursively, we could repeat this procedure for the subfamilies, and find many separable members in each of them.

It may happen that already at the first non-trivial cut we are forced to destroy (i.e. cut through) a large fraction of the members of $\mathcal{F}$. Consider the following family. Let $p_1, p_2, \ldots, p_{2n}$ be the vertices of a regular $2n$-gon of diameter 1 in the plane. For every i, $1 \leq i \leq n$, let F_i be the segment of length K starting at p_{2i} and passing through p_{2i-1}. If K is sufficiently large, then any straight line ℓ with the property that both half-planes bounded by ℓ fully contain at least one member of $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ must cut through roughly half of the F_i-s (see Figure 3). On the other hand, having performed such a cut, the remaining members are separable. A similar construction was described by R. Hope [H84] (see also [HK90]).

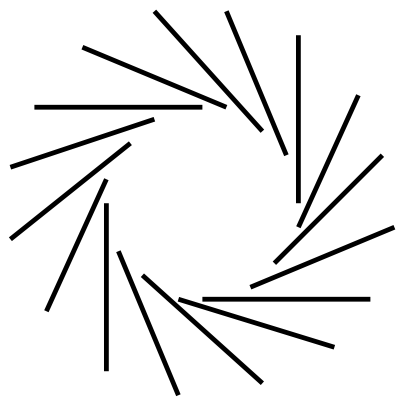


Figure 3.

There is a more serious difficulty with the above approach. It is not hard to modify the previous construction so that there is *no* straight-line which has at least two members on both of its sides. (See [T79], [PT00].)

6.2 In the proof of Theorem 4.1, it seems tempting to replace the application of Theorem 1.5 by an iterative argument. The difficulty is that after the first round of cuts we can no longer guarantee that the ratio between the sizes of the sets is bounded from above in terms of the number of sets in the family. In case we could use the techniques of this proof without having such a bound, we could iterate our procedure and obtain a larger separable set.

6.3 We say that a family of pairwise disjoint sets in the plane is *strongly separable*, if any two members can be separated from each other by a straight line which does not cut through any of the remaining sets. It is not true that every large family of pairwise disjoint compact convex sets in the plane has many strongly separable members. Indeed, it is not hard to construct a family of infinitely many pairwise disjoint straight-line segments in the plane, no three of which are strongly separable. For some positive results, consult [PT00].

6.4 As mentioned in the Introduction, we conjecture that the worst possible constructions for separability (i.e. those which have the smallest number of separable members) can be realized by segments.

Another optimization problem for plane convex bodies whose solution is probably also realizable by straight-line segments was studied in [G94],[CP98]. It is a common feature of these problems that all non-trivial results known for them were obtained by introducing certain partial orders on the family of convex bodies and then applying some form of Dilworth's theorem.

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