Monochromatic empty triangles in two-colored point sets

János Pach*and Géza Tóth†
Rényi Institute, Budapest

Abstract

Improving a result of Aichholzer et. al., we show that there exists a constant $c > 0$ satisfying the following condition. Any two-colored set of n points in general position in the plane has at least $cn^{4/3}$ triples of the same color such that the triangles spanned by them contain no element of the set in their interiors.

1 Introduction

Let P be a set of points in the plane in *general position*, that is, assume that if no *three* elements of P are on a line. A subset of P is said to be in *convex position* if it is the vertex set of a convex polygon. According to a classical result of Erdős and Szekeres [ErSz35], for every integer $k > 3$ there exists an $n(k)$ such that any set P of at least $n(k)$ points in general position in the plane has a k-element subset in convex position. For a long time it was conjectured that if P sufficiently large, then it must also contain the vertex set of an *empty* convex k-gon, that is, one that has no element of P in its interior. This statement can be easily verified for $k \leq 5$. In 1983, Horton [Ho83] surprised the combinatorics community by constructing arbitrarily large point sets with no empty convex *heptagon*. It took another quarter of a century to verify the conjecture for *hexagons* [Ge08, Ni07].

Some colored variants of the Erdős-Szekeres problem were considered by Devillers, Hurtado, Károlyi, and Seara [DeH03]. In particular, it is easy to see that any 2-colored point set of size *ten* in general position in the plane has a *monochromatic* triple inducing an empty triangle. It follows, for example, that any set of n points spans at least $\lfloor (n-1)/9 \rfloor$ monochromatic empty triangles. It is not easy to see that the number of such triangles must be superlinear in n. This has been proved recently by Aichholzer, Fabila-Monroy, Flores-Penaloza, Hackl, Huemer, and Urrutia [AiF08], who established a lower bound of $cn^{5/4}$. Here we modify some of their ideas to obtain a somewhat better bound.

Theorem. *Any two-colored set of n points in general position in the plane spans at least $cn^{4/3}$ monochromatic empty triangles, where $c > 0$ is an absolute constant.*

A number of related questions for colored point sets are listed in [BrM05, KaKa03].

*Supported by NSF grant CCF-05-14079 and grants from NSA, PSC-CUNY, Hungarian Research Foundation, and BSF.

†Supported by OTKA-K-60427.

2 Proof of Theorem

It is assumed throughout this note that the point set we consider is in general position. To make this note self-contained, we include the short proofs of the following two lemmas taken from the paper of Aichholzer *et al.* [AiF08].

Order Lemma ([AiF08]). *Let $P_1P_2P_3$ be a triangle containing the points $Q_1, Q_2, \ldots, Q_m$ in its interior. Then the set $\{P_1,\ P_2,\ P_3,\ Q_1,\ \ldots,\ Q_m\}$ can be triangulated so that at least $m + \lceil\sqrt{m}\rceil + 2$ triangles have P_1, P_2, or P_3 as one of their vertices.*

Proof. Define a partial order $\prec$ on points $Q_1, Q_2, \ldots Q_m$ as follows. We say that $Q_i \prec Q_j$ if and only if triangle $Q_jP_1P_2$ contains Q_i. By Dilworth's theorem, there exists (i) a chain or (ii) an antichain of size $m' = \lceil\sqrt{m}\rceil$.

Suppose first that there is a chain of length m'. Assume without loss of generality that $Q_1 \prec \ldots \prec Q_{m'}$ is such a chain. Add all edges Q_iQ_{i+1}, $i = 1, \ldots m' - 1$ and all edges Q_iP_1, Q_iP_2, $i = 1, \ldots m'$. Together with edge P_1P_2, now we have a triangulation of the set $\{P_1, P_2, Q_1, \ldots, Q_{m'}\}$. Each of the remaining points $Q_{m'+1}, \ldots, Q_m$ can be connected to P_1 or to P_2 by an edge not crossing any of the previously selected edges. Connect those "visible" from P_1 to P_1, and the others to P_2, and include the edges P_1P_3, and P_2P_3. We have obtained a set of noncrossing edges (a planar graph) such that the total degree of P_1 and P_2 is $m + m' + 4$. Extend this graph to a triangulation of the set $\{P_1, P_2, P_3, Q_1, \ldots, Q_m\}$. At least $m + m' + 2$ triangles have P_1 or P_2 as one of their vertices, so in this case we are done.

Suppose now that, for example, $Q_1, \ldots, Q_{m'}$ is an antichain of size m'. Then none of the $\binom{m'}{2}$ lines induced by these points intersects the segment P_1P_2. Thus, all of them must cross both P_1P_3 and P_2P_3. Consequently, for any $1 \leq i < j \leq m'$, either $P_1P_3Q_i$ contains Q_j or $P_1P_3Q_j$ contains Q_i. Now we can finish the argument as in the first case, except that the roles of P_2 and P_3 must be interchanged. □

Discrepancy Lemma ([AiF08]). *Any set of n blue and $n + k$ red points in general position in the plane spans at least $(n + k)(k - 2)/3$ monochromatic empty triangles.*

Proof. Let P be one of the red points. Let $P_1, \ldots, P_{n+k-1} = P_0$ denote the other red points in the order of visibility from P.

Each angle $\langle P_iPP_{i+1}\rangle$ is smaller than π, with at most one possible exception, $\langle P_0PP_1\rangle$, say. Therefore, the interiors of the triangles P_1PP_1, P_2PP_3, $\ldots$, $P_{n+k-2}PP_{n+k-1}$ are pairwise disjoint. Since at most n of them can contain a blue point, at least $k - 2$ of them must be empty. Repeating this argument for each red point P, we obtain at least $(n + k)(k - 2)$ empty red triangles, each of which is counted at most *three* times. □

Return now to the proof of the Theorem. Given any set S of $r(S)$ red and $b(S)$ blue points, define the *discrepancy* of S as

$$d(S) := |r(S) - b(S)|.$$

Let S be a two-colored set of n points in general position, and suppose, for simplicity, that $n \geq 1000$. We call a point $P \in S$ *rich* if there are at least $\sqrt[3]{n}$ empty monochromatic triangles adjacent to P. The following algorithm proves the Theorem by finding at least $n/5$ rich points.

Algorithm Find-Rich-Points(S)

Step 0. If $d(S) \geq \sqrt[3]{n}/100$, then, by the Discrepancy Lemma, we find $\Omega(n^{4/3})$ monochromatic empty triangles. Assume that $d(S) < \sqrt[3]{n}/100$. It follows that $b = b(S) > n/2 - \sqrt[3]{n}/200$ and $r = r(S) > n/2 - \sqrt[3]{n}/200$. Set $i = 1$ and $S_1 = S$.

Step i. It follows by induction on i that $b(S_i) = b(S_{i-1}) - 1$, for $i > 1$, so that we have $b = b(S_i) > n/2 - \sqrt[3]{n}/200 - i$, for all $i \geq 1$. Assuming that our algorithm stops before finding at least $n/5$ rich points, we have $i \leq n/5$.

Take the convex hull of S_i. Remove all red points from its boundary and take the convex hull of the remaining set. Remove again all red points from the boundary and continue until we obtain a set S' whose convex hull contains only blue points. So far we have not removed any blue point, so that we have $b(S') = b(S_i)$. If $d(S') \geq \sqrt[3]{n}/100$, then Stop and observe that we are done by the Discrepancy Lemma. So we may and will assume that $d(S') < \sqrt[3]{n}/100$. It follows that $r = r(S') \geq b(S') - d(S') > n/2 - 3\sqrt[3]{n}/200 - i > n/4$. If the convex hull of S' has at least $\sqrt[3]{n}/50$ points, remove them, and denote the resulting set by S''. Since $d(S') \leq \sqrt[3]{n}/100$ and all points that have been removed in the last step were of the same color (blue), we have $d(S'') \geq \sqrt[3]{n}/100$. Taking into account that $|S''| \geq r(S') > n/4$, we are done by the Discrepancy Lemma, so we can Stop. Therefore, we can assume that there are m points on the boundary of the convex hull of S', all of them blue, for some $m \leq \sqrt[3]{n}/50$. Let $P_1, P_2, \ldots, P_m$ denote these points, in clockwise order. Triangulate the convex hull of S' by adding the diagonals P_1P_j, for $j = 2, \ldots m-2$. Let T_j denote the triangle $P_1P_{j+1}P_{j+2}$, and let b_j and r_j be the number of blue and red points of S' lying in the interior of T_j ($j = 1, \ldots, m-2$).

Suppose that $|b_j - r_j| > \sqrt[3]{n}/50$, for some j. At least one of the regions T_j, $T_1 \cup T_2 \cdots \cup T_{j-1}$, and $T_{j+1} \cup \cdots \cup T_{m-2}$ contains at least $(r(S') + b(S') - m)/3 \geq (2b(S') + d(S') - m)/3 = (2b(S_i) + d(S') - m)/3 \geq n/6$ points. If T_j is such a region, then we can apply the Discrepancy Lemma for the points inside T_j and we are done. If $T_1 \cup T_2 \cdots \cup T_{j-1}$ contains at least $n/6$ points, then either $S' \cap (T_1 \cup T_2 \cdots \cup T_{j-1})$, or $S' \cap (T_1 \cup T_2 \cdots \cup T_{j-1} \cup T_j)$ has discrepancy at least $\sqrt[3]{n}/100$, and again we are done and we Stop.

Therefore, we can assume that $|b_j - r_j| \leq \sqrt[3]{n}/50$, for every $j = 1, \ldots, m-2$. Since $\sum_{j=1}^{m-2} b_j = b(S') - m = b(S_i) - m \geq n/4$, there exists a j such that $b_j \geq n/(4m) \geq 50n^{2/3}/4$ and, by our assumption, $r_j \leq b_j + \sqrt[3]{n}/50$. By the Order Lemma, we can triangulate the blue points in T_j, including the vertices of T_j, such that at least $b_j + \sqrt{b_j} > b_j + 7\sqrt[3]{n}/2$ triangles are adjacent to one of the vertices of T_j. At least $7\sqrt[3]{n}/2 - \sqrt[3]{n}/50 > 3\sqrt[3]{n}$ of these triangles does not contain a red point, and at least one-third of these empty triangles shares the same vertex of T_j, denoted by P. Thus, we have found at least $\sqrt[3]{n}$ empty triangles incident to the same vertex P, which is therefore a Rich Point. If $i \geq n/5$, then Stop. Otherwise, let $S_{i+1} = S_i \setminus \{P\}$, and set $i := i + 1$.

Summarizing: Algorithm Find-Rich-Points(S) either stopped at Step i for some $i \leq n/5$, or at Step $\lceil n/5 \rceil$. In the first case, it stopped because we applied the Discrepancy Lemma to find $\Omega(n^{4/3})$ empty monochromatic triangles. In the second case, we found at least $n/5$ rich points, and hence at least $n^{4/3}/15$ empty monochromatic triangles. This concludes the proof of the Theorem. □

Note that it is perfectly possible that any two-colored set of n points in general position in the plane spans at least a quadratic number of monochromatic empty triangles, that is, the lower bound $cn^{4/3}$ in the Theorem can be replaced by cn^2, for a suitable constant $c > 0$. Of course, the order of magnitude of this bound would be best possible.

References

[AiF08] O. Aichholzer, R. Fabila-Monroy, D. Flores-Penaloza, T. Hackl, C. Huemer, and J. Urrutia, Monochromatic empty triangles, preprint, 2008.

[BrM05] P. Brass, W. Moser, and J. Pach: *Research Problems in Discrete Geometry*, Springer, New York, 2005.

[DeH03] O. Devillers, F. Hurtado, Gy. Károlyi, and C. Seara: Chromatic variants of the Erdős-Szekeres theorem on points in convex position, *Comput. Geom.* **26** (2003), no. 3, 193–208.

[ErSz35] P. Erdős and G. Szekeres: A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[Ge08] T. Gerken: Empty convex hexagons in planar point sets, *Discrete Comput. Geom.* **39** (2008), no. 1-3, 239–272.

[Ho83] J. D. Horton: Sets with no empty convex 7-gons, *Canad. Math. Bull.* **26** (1983), no. 4, 482–484.

[KaKa03] A. Kaneko and M. Kano: Discrete geometry on red and blue points in the plane—a survey, in: *Discrete and Computational Geometry, Algorithms Combin.*, 25, Springer, Berlin, 2003, 551–570.

[Ni07] C. M. Nicolás: The empty hexagon theorem, *Discrete Comput. Geom.* **38** (2007), no. 2, 389–397.