Decomposition of a cube into nearly equal smaller cubes

Peter Frankl [*] Amram Meir[†] János Pach[‡]

Abstract

Let d be a fixed positive integer and let $\varepsilon > 0$. It is shown that for every sufficiently large $n \geq n_0(d, \varepsilon)$, the d-dimensional unit cube can be decomposed into exactly n smaller cubes such that the ratio of the side length of the largest cube to the side length of the smallest one is at most $1 + \varepsilon$. Moreover, for every $n \geq n_0$, there is a decomposition with the required properties, using cubes of at most $d+2$ different side lengths. If we drop the condition that the side lengths of the cubes must be roughly equal, it is sufficient to use cubes of *three* different sizes.

1 Introduction

It was shown by Dehn [3] that, for $d \geq 2$, in any decomposition (tiling) of the d-dimensional unit cube into finitely many smaller cubes, the side length of every participating cube must be rational. Sprague [9] proved that in the plane there are infinitely many decompositions consisting of pairwise incongruent squares. Such a decomposition is called *perfect.* Brooks, Smith, Stone, and Tutte [1] developed a method to generate all perfect decompositions of squares, by reformulating the problem as a problem for flows in electrical networks. Duijvestijn [4] discovered the unique perfect decomposition of a square into the smallest number of squares: it consists of 21

[*]Rényi Institute, Hungarian Academy of Sciences, H–1364 Budapest, POB 127, Hungary. Email: peter.frankl@gmail.com

[†]York University, 1121 Steeles Ave West, apt. PH-206, Toronto, Ontario, M2R-3W7, Canada. Email: ameir@rogers.com

[‡]Rényi Institute and EPFL, Station 8, CH–1014 Lausanne, Switzerland. Email: pach@cims.nyu.edu. Supported by Swiss National Science Foundation Grants 200020-144531 and 200021-137574.

pieces. It is not hard to see that in 3 and higher dimensions no perfect tilings exist [1].

Fine and Niven [6] and Hadwiger raised the problem of characterizing, for a fixed $d \geq 2$, the set S_d of all integers n such that the d-dimensional unit cube can be decomposed into n not necessarily pairwise incongruent cubes. Obviously, $i^d \in S_d$ for every positive integer i. Hadwiger observed that no positive integer smaller than 2^d, or larger than 2^d but smaller than $2^d + 2^{d-1}$, belongs to S_d. On the other hand, Plüss [8] and Meier [7] showed that for any $d \geq 2$, there exists $n_0(d)$ such that all $n \geq n_0(d)$ belong to S_d. It is known that $n_0(2) = 6$ and it is conjectured that $n_0(3) = 48$ (see [2], problems C2, C5). The best known general upper bound $n_0(d) \leq (2d-2)((d+1)^d - 2) - 1$ is due to Erdős [5]. It is conjectured that this can be improved to $n_0(d) \leq c^d$ for an absolute constant c.

To show that $n_0(2) \leq 6$, consider arbitrary decompositions of the square into $6, 7$, and 8 smaller squares; see Fig. 1.

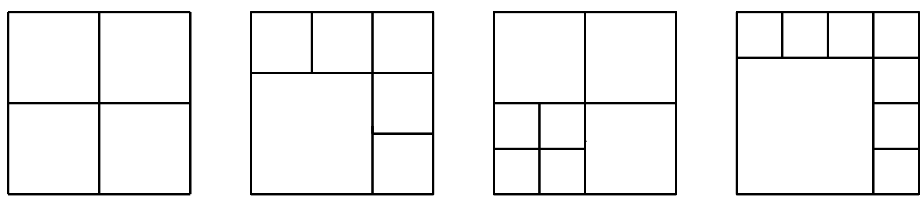

Figure 1: Decompositions of the square into 4, 6, 7, and 8 squares. There exists no decomposition into 5 squares.

Notice that from any decomposition into n squares, one can easily obtain a decomposition into $n+3$ squares by replacing one of the squares, Q, with *four* others whose side lengths are half of the side length of Q. If we are careless, during this process we may create squares of many different sizes. In particular, for most values of n, the ratio of the side length of the largest square of the decomposition to the the side length of the smallest square is at least 2.

Amram Meir asked many years ago whether for any $d \geq 2, \varepsilon > 0$, and for every sufficiently large $n \geq n_0(d, \varepsilon)$, there exists a decomposition of a d-dimensional cube into n smaller cubes such that the above ratio is smaller than $1+\varepsilon$. The aim of this paper is to answer this question in the affirmative. In Section 2, we give a simple construction in the plane, which does not seem to generalize to higher dimensions.

Theorem 1. *For any integer $n \geq 6$ that is not a square number, there exists a tiling of a square with smaller squares of two different sizes such that the ratio of their side length tends to 1 as $n \to \infty$.*

Of course, if n is a square number, then any square can be decomposed into precisely n smaller squares of the same size.

In Section 3, we review some elementary number-theoretic facts needed for the proof of the following theorem, which will be established in Section 4.

Theorem 2. *For any integer $d \geq 2$ and any $\varepsilon > 0$, there exists $n_0 = n_0(d, \varepsilon)$ with the following property. For every $n \geq n_0$, the d-dimensional unit cube can be decomposed into n smaller cubes such that the ratio of the side length of the largest subcube to the side length of the smallest one is at most $1 + \varepsilon$. Moreover, for every $n \geq n_0$, there is a decomposition with the required properties, using subcubes of at most $d + 2$ different side lengths.*

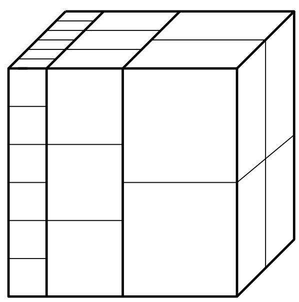

Figure 2: A decomposition of the 3-dimensional cube into 49 cubes.

The idea of the proof of Theorem 2 is the following. If n is of the form a^{3d} for some positive integer a, then we first divide the unit cube into a^{2d} *small* cubes of side length $1/a^2$. In the second step, we subdivide each small cube into a^d even smaller subcubes of side length $1/a^3$, and we are done. We call these subcubes *tiny.* If n is not of this special form, we have $a^{3d} < n < a^{3(d+1)}$ for some a. In this case, after the first step, the number of tiny cubes will be smaller than n. Therefore, in the second step, we have to subdivide some of the small cubes into *slightly more* than a^d equal subcubes. Unfortunately, this simple strategy does not necessarily work: the number of small cubes that need to be subdivided into more than a^d pieces may exceed a^{2d}, the total number of small cubes produced in the first step. To overcome this difficulty, in addition, a certain number of small cubes will be subdivided into *fewer* than a^d tiny cubes. The details are worked out in Section 4.

In the last section, we prove that for any $d \geq 3$ and any sufficiently large n depending on d, it is possible to tile a cube with precisely n smaller cubes of at most *three* different sizes; see Theorem 5.1. We close the paper with a few open problems.

2 The planar case – Proof of Theorem 1

For convenience, we start with a simple observation that allows us to disregard small values of n and to suppose in the rest of the argument that n is sufficiently large.

Lemma 2.1 *For every integer $n \geq 6$, a square can be tiled with n smaller squares of at most* two *different sizes.*

Proof: Consider the first, second, and last tilings depicted in Fig. 1. They can be generalized as follows. For every integer $k \geq 1$, take the unit square and extend it to a square S_k of side length $1 + \frac{1}{k}$ by adding $2k+1$ squares of side length $\frac{1}{k}$ along its upper side and right side. We obtain a tiling with $2k+2$ squares of two different sizes, unless $k = 1$ and all 4 squares are of the same size.

By dividing the original unit square of this tiling into 4 equal squares, we obtain a tiling of S_k with $2k+5$ squares of side lengths $\frac{1}{2}$ and $\frac{1}{k}$. As k runs through all positive integers, we obtain tilings with n squares, for $n = 4$ and all $n \geq 6$. $\square$

Next, we write every positive integer $n > 36$ in a special form.

Claim 2.2 *Let n be a positive integer satisfying $a^2 < n < (a+1)^2$ for some positive integer a. Then there exists an integer b such that $a < b \leq 2a$ and*

(i) either $n = a^2 + b$

(ii) or $n = (a+1)^2 - b$.

Proof: If $n \leq a^2 + a$, then (ii) holds. If $n > a^2 + a$, then (i) is true. $\square$

Claim 2.3 *Let a and b be positive integers with $b > a \geq 6$. Then there exists an integer m with $\frac{b-2}{4} \leq m \leq \frac{b-1}{2}$ such that precisely one of the following conditions is satisfied.*

(i) $b = 2m + 1 = (m+1)^2 - m^2$,
(ii) $b = 4m = (m+1)^2 - (m-1)^2$,
(iii) $b = 4m + 2 = 2(m+1)^2 - 2m^2$. □

Proof of Theorem 1: In view of Lemma 2.1, it is sufficient to prove Theorem 1 for sufficiently large n. From now on, suppose that $n \geq 36$. We can also assume that n is not a perfect square, otherwise, the statement is trivially true. Then we have $a^2 < n < (a+1)^2$, for some $a \geq 6$. Let b be the integer whose existence is guaranteed by Claim 2.2, and write b, as in Claim 2.3, in the form (i), (ii), or (iii). Note that a, b, and m are uniquely determined, and we clearly have $b > a > \sqrt{n} - 1 \to \infty$ and $m \geq \frac{b-2}{4} \to \infty$, as $n \to \infty$.

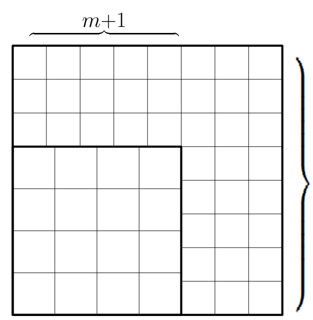

Figure 3: Illustration to **Case 1**. The parameters are $n = 55$, $a = 7$, $b = 9$, and $m = 4$, so that we have $n = (a+1)^2 - (m+1)^2 + m^2$.

If b satisfies conditions (i) or (ii) in Claim 2.3, we obtain n as an expression consisting of three squares, two with positive signs and one with a negative sign. More precisely, we have

$$n = p^2 - q^2 + r^2 \quad \text{with} \quad p > q, \tag{1}$$

where $p \in \{a, a+1\}$, and $q, r \in \{m-1, m, m+1\}$.

If b satisfies condition (iii) in Claim 2.3, we get

$$n = p^2 - 2q^2 + 2r^2 \quad \text{with} \quad p \geq 2q, \tag{2}$$

where $p \in \{a, a+1\}$, and $q, r \in \{m, m+1\}$.

We construct slightly different tilings in the above two cases.

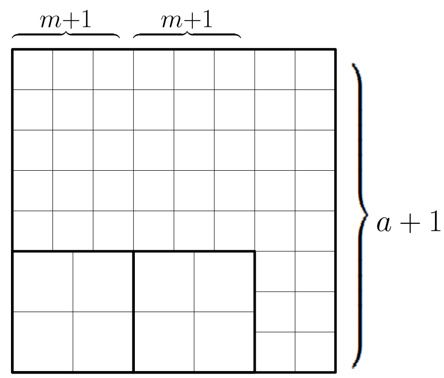

Figure 4: Illustration to **Case 2**. The parameters $n = 54$, $a = 7$, $b = 10$, and $m = 2$, so that we have $n = (a+1)^2 - 2(m+1)^2 + 2m^2$.

Case 1: If (1) holds, take a $p \times p$ square divided into unit squares, and replace a $q \times q$ part of it by a tiling consisting of r^2 squares of side length $\frac{r}{q}$. See Fig. 3.
Case 2: If (2) holds, take a $p \times p$ square divided into unit squares, choose two disjoint $q \times q$ subsquares in it, and tile each of them with r^2 squares of side length $\frac{r}{q}$. See Fig. 4.

In both cases, we obtain a tiling of a large square that consists of precisely n smaller squares such that the ratio of their side lengths is at most $\frac{m+1}{m-1} = 1 + O(\frac{1}{\sqrt{n}})$. This completes the proof of Theorem 1. $\square$

Problem 2.4 *Let $\rho(n)$ denote the smallest ratio of the maximum side length of a square to the minimum side length of a square over all decompositions of a square into n smaller ones. Describe the asymptotic behavior of the function*

$$\limsup_{n\to\infty} \rho(n) - 1.$$

It follows from the above proof that $\rho(n) - 1 = O(\frac{1}{\sqrt{n}})$, but we have no good lower bound.

3 Number-theoretic preliminaries

Before turning to the proof of Theorem 2, in this short section we collect and prove some simple facts we need from elementary number theory.

Lemma 3.1 *Let $a_1 < \ldots < a_r$ and $b_1 < \ldots < b_s$ be positive integers whose greatest common divisor is* 1.

(i) For every integer k, there exist integers $x_1, \ldots, x_r, y_1, \ldots, y_s \geq 0$ with

$$\sum_{i=1}^{r} x_i a_i - \sum_{j=1}^{s} y_j b_j = k.$$

(ii) Moreover, we can assume that $\max_i x_i < b_s$ or $\max_j y_j < a_r$.

Proof: Part (i) goes back to Euclid. As for part (ii), if $x_i \geq b_s$ and $y_j \geq a_r$ for some i and j, then we can replace x_i with $x_i - b_j$ and y_j with $y_j - a_i$, and (i) continues to hold. Repeating this step, if necessary, (ii) follows. □

In the last section, we will also use the following well known statement, due to Sylvester [10].

Lemma 3.2 *Let a_1 and a_2 be positive integers whose greatest common divisor is* 1. *Then for every integer $k \geq (a_1 - 1)(a_2 - 1)$, there exist integers $x_1, x_2 \geq 0$ with $k = x_1 a_1 + x_2 a_2$.* □

Lemma 3.3 *Let $d \geq 2$ be an integer and p a prime. Then, for every fixed integer m, there exists t, $1 \leq t \leq d$ such that p does not divide $(m+t)^d - m^d$.*

Proof: Consider the polynomial $q(x) = (x+1)^d - x^d$ of degree $d-1$. We have $q(p) \equiv 1 \mod p$. Therefore, $q(x)$ is not the zero-polynomial and it has at most $d-1$ roots over $\mathrm{GF}(p)$. Consequently, at least one of the numbers $q(m), q(m+1), \ldots, q(m+d-1)$ is not divisible by p. If p does not divide $q(m)$, we are done. Otherwise, suppose that p divides $q(m), \ldots, q(m+t-2)$, but does not divide $q(m+t-1)$ for some $t, 2 \leq t \leq d$. Then we have

$$(m+t)^d - m^d = \sum_{i=0}^{t-1} q(m+i) \equiv q(m+t-1) \not\equiv 0 \mod p,$$

as required. □

Corollary 3.4 *For any integers $d, m > 0$, the greatest common divisor of the numbers $(m+1)^d - m^d, (m+2)^d - m^d, \ldots, (m+d)^d - m^d$ is* 1. □

4 Proof of Theorem 2

Let ε be a (small) positive number, which will be fixed throughout this section.

Suppose first that $n = a^{3d}$ for a positive integer a. Dividing the unit cube into a^{2d} *small* cubes of side length $1/a^2$, and then each small cube into a^d *tiny* subcubes of the same size, we are done.

If n is not of the above special form, we have $a^{3d} < n < (a+1)^{3d}$ for some integer $a > 0$. Suppose that n is so large that we have $a > d(d+1)$ and

$$(a-1)(1+\varepsilon) > a + d \tag{3}$$

Using the assumption $a > d(d+1) \geq 6$, we have that $a^{2d}(a+4)^d > (a+1)^{3d}$. Thus, there exists an integer c, $0 \leq c \leq 3$, with

$$a^{2d}(a+c)^d \leq n < a^{2d}(a+c+1)^d. \tag{4}$$

Let $m := a + c$.

We now construct a tiling of the unit cube with n smaller cubes, each of side length $\frac{1}{a^2(m+i)}$ for some i, $-1 \leq i \leq d$. Now these smaller cubes will be called *tiny*.

In the first step, divide the unit cube into a^{2d} *small* cubes of side length $1/a^2$. If we subdivided each small cube into m^d subcubes of side length $1/(a^2 m)$, we would obtain only $a^{2d}m^d \leq n$ tiny cubes of the same size. In order to obtain precisely n tiny cubes, we need to increase their number by

$$k := n - a^{2d}m^d.$$

To achieve this, for $i = 1, 2, \ldots, d$, we pick x_i small cubes, and instead of subdividing them into m^d tiny subcubes, we subdivide them into $(m+i)^d$ ones. This results in an increase of $\sum_{i=1}^{d} x_i a_i$, where

$$a_i = (m+i)^d - m^d.$$

However, if k is relatively small (for example, if $k = 1$), we may not be able to write k in the form $\sum_{i=1}^{d} x_i a_i$. Therefore, we also select y_1 small cubes, different from the previously picked ones, and instead of subdividing them into m^d pieces, we subdivide them into only $(m-1)^d$ tiny subcubes of the same size. This will reduce the number of tiny subcubes by $y_1 b_1$, where

$$b_1 = m^d - (m-1)^d.$$

By Corollary 3.4, the greatest common divisor of $a_1, \ldots, a_d, b_1$ is 1. Applying Lemma 3.1 with $r = d, s = 1$, we can find nonnegative numbers $x_1, \ldots, x_d, y_1$ such that

$$\sum_{i=1}^{d} x_i a_i - y_1 b_1 = k, \tag{5}$$

as required. If, in addition, we manage to guarantee that

$$x_1 + \ldots + x_d + y_1 \leq a^{2d}, \tag{6}$$

then we are done, because there are sufficiently many small cubes in which the required replacements can be performed.

In what follows, we show how to find a solution of (5), for which condition (6) is satisfied. Start with any solution of (5) and, as long as possible, repeat the following two steps, producing other solutions.

1. If $x_i \geq b_1$ for some $i,\ 1 \leq i \leq d$, and $y_1 \geq a_d$, then replace x_i by $x_i - b_1$ and y_1 by $y_1 - a_d$.
2. If $x_i \geq a_d$ for some $i,\ 1 \leq i < d$, then replace x_i by $x_i - a_d$ and x_d by $x_d + a_i$.

Both of these operations, independently, can be performed at most a bounded number of times. Thus, the above procedure will terminate after a finite number of steps. We claim that the resulting solution satisfies (6).

It follows from $m \geq a > d(d+1)$ that

$$a_d = (m+d)^d - m^d = \left((1 + \frac{d}{m})^d - 1\right) m^d < (e-1)m^d. \tag{7}$$

We also have

$$b_1 = m^d - (m-1)^d \leq (a+3)^d - (a+2)^d < (e-1)a^d \tag{8}$$

and

$$m^d \leq (a+3)^d = \left(1 + \frac{3}{a}\right)^d a^d < \left(1 + \frac{3}{d(d+1)}\right)^d a^d < ea^d. \tag{9}$$

We distinguish two cases.

Case 1: $y_1 \geq a_d$

Since we cannot perform step 1, we have $x_i < b_1$ for all i ($1 \le i \le d$). From (5), we obtain

$$y_1 b_1 < \sum_{i=1}^{d} x_i a_i < \sum_{i=1}^{d} b_1 a_i \le d b_1 a_d.$$

In view of (7) and (9), this yields that

$$y_1 < d a_d < d(e-1)m^d < de(e-1)a^d.$$

Taking (8) into account, in this case we have

$$x_1 + \ldots + x_d < d b_1 < d(e-1)a^d.$$

Combining the last two inequalities, we get

$$x_1 + \ldots + x_d + y_1 < d(e-1)a^d + de(e-1)a^d = d(e^2-1)a^d < a^{2d},$$

so (6) holds, as required.

Case 2: $y_1 < a_d$

Since we cannot perform step 2 of the algorithm, we have $x_i < a_d$ for all i, $1 \le i < d$. Now, using (7) and (9), we obtain

$$x_1 + \ldots + x_{d-1} + y_1 < d a_d < d(e-1)m^d < de(e-1)a^d. \tag{10}$$

It remains to bound x_d from above. From (5), we have $x_d a_d \le k + y_1 b_1$. Since $k = n - a^{2d} m^d$, in view of (4), we get

$$x_d a_d < a^{2d}((m+1)^d - m^d) + y_1 b_1 < a^{2d}((m+1)^d - m^d) + a_d b_1.$$

Hence,

$$x_d - b_1 < a^{2d} \frac{(m+1)^d - m^d}{a_d} = a^{2d} \frac{(m+1)^d - m^d}{(m+d)^d - m^d} < a^{2d} \frac{1}{d}. \tag{11}$$

The last inequality follows from the fact that

$$(m+d)^d - m^d = \sum_{i=0}^{d-1} ((n+i+1)^d - (n+i)^d) > d((n+1)^d - n^d).$$

Adding (10), (11), and (8), and using that $a > d(d+1)$, we conclude that

$$\begin{aligned} x_1 + \ldots + x_d + y_1 &< de(e-1)a^d + a^{2d}\frac{1}{d} + (e-1)a^d \\ &= a^{2d}\left(\frac{1}{d} + \frac{(de+1)(e-1)}{a^d}\right) < a^{2d}. \end{aligned}$$

Thus, in both cases, (6) is true. This completes the proof of the Theorem 2. □

5 Concluding remarks, open problems

The proof in the previous section shows that, for any sufficiently large n, the d-dimensional unit cube can be tiled with n smaller cubes having at most $d+2$ different sizes. Moreover, we can even require these smaller cubes to be of roughly the same size. Lemma 2.1 states that for $d=2$, *two* different sizes suffice. For any d larger than 2, if we do drop the condition that the ratio of the sizes of the largest and smallest cubes must tend to 1, one can decompose a cube into smaller cubes of only *three* different sizes.

Theorem 5.1 *For any $d \geq 3$, there is an $n_0 = n_0(d)$ with the following property. For any integer $n \geq n_0(d)$, there exists a tiling of a cube with precisely n smaller cubes, each of side length 1, $\frac{1}{2}$, or $\frac{1}{2^d-1}$.*

Proof: Since $(2^d-1)^d$ is a multiple of 2^d-1, the numbers $(2^d-1)^d-1$ and 2^d-1 are relatively prime. Thus, we can apply Lemma 3.2 with $a_1 = 2^d-1$ and $a_2 = (2^d-1)^d-1$. It implies that every integer $k \geq 2^{(d+1)d}$ can be expressed as

$$k = x_1(2^d-1) + x_2((2^d-1)^d-1), \tag{12}$$

for suitable integers $x_1, x_2 \geq 0$.

Suppose that $n > 2^{(d+3)d}$. Then we can choose an integer $a \geq 2^{d+3}$ such that $a^d \leq n < (a+1)^d$. Consider an $(a-1)\times(a-1)$ cube C of side length $a-1$, and decompose it into $(a-1)^d$ unit cubes. Set $k := n-(a-1)^d$. Observe that

$$k \geq a^d - (a-1)^d \geq (2^{d+3})^d - (2^{d+3}-1)^d \geq d(2^{d+3}-1)^{d-1} \geq 2^{(d+1)d}.$$

Therefore, we can write k in the form (12), where x_1 and x_2 are nonnegative integers and, clearly,

$$x_1 + x_2 \leq \frac{k}{2^d-1} = \frac{n-(a-1)^d}{2^d-1} < \frac{(a+1)^d-(a-1)^d}{2^d-1} < (a-1)^d. \tag{13}$$

When we subdivide a unit cube in C into cubes of side length $\frac{1}{2}$, the number of cubes increases by 2^d-1. When we subdivide a unit cube in C into cubes of side length $\frac{1}{2^d-1}$, the number of cubes in the tiling increases by $(2^d-1)^d-1$. According to (13), there are enough unit cubes in C to make x_1 subdivisions of the first kind and x_2 subdivisions of the second kind. The resulting tiling consists of precisely n cubes, as required. $\square$

The last theorem can be generalized in possibly two different ways.

Problem 5.2 *Can one find, for every sufficiently large positive integer n,* two 3*-dimensional cubes with side lengths smaller than* 1 *such that the unit cube can be tiled with their congruent copies?*

Problem 5.3 *Does there exist, for every sufficiently large positive integer n, a tiling of the unit cube with n smaller cubes having at most* three *(or at most* two*) different side lengths so that the ratios of these side length tend to* 1*, as $n \to \infty$?*

Theorem 2 shows that in 3-dimensional space one can find tilings consisting of copies of *four* different cubes such that the ratios of their side lengths tend to 1. However, in higher dimensions the number of different cubes used in our construction grows.

Problem 5.4 *Does there exist an absolute constant k such that for any $d \geq 3$ there exists $n_0(d)$ satisfying the following property? For every $n \geq n_0(d)$, there is a tiling of a cube in $\mathbb{R}^d$ with precisely n cubes of at most k different side lengths.*

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