Canonical theorems for convex sets

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Abstract

Let $\mathcal{F}$ be a family of pairwise disjoint compact convex sets in the plane such that none of them is contained in the convex hull of two others, and let r be a positive integer. We show that $\mathcal{F}$ has r disjoint $\lfloor c_r n \rfloor$-membered subfamilies $\mathcal{F}_i$ $(1 \leq i \leq r)$ such that no matter how we pick one element F_i from each $\mathcal{F}_i$, they are in convex position, i.e., every F_i appears on the boundary of the convex hull of $\cup_{i=1}^{r} F_i$. (Here c_r is a positive constant depending only on r.) This generalizes and sharpens some results of Erdős–Szekeres, Bisztriczky–Fejes Tóth, Bárány–Valtr, and others.

1 Introduction

In their classical paper written in 1935, Erdős and Szekeres [ES1], [E] proved that for every $r \geq 3$, there exists an integer $f(r)$ such that any set of at least $f(r)$ points in the plane has r elements in convex position. This result has inspired a lot of research in combinatorial geometry and in Ramsey theory (see e.g. [BDV], [GRS], [H], [PA], [TV], [V]).

It follows that if n is much larger than $f(r)$, then every n-element point set P contains many r-tuples in convex position. For instance, Solymosi [S] showed that for a suitable constant $c_r > 0$, one can select a sequence of $c_r n$ distinct elements from P, whose any r consecutive members are in convex position. In the case $r = 4$, Nielsen [N] and, in general, Bárány and Valtr [BV] proved the following stronger result.

Theorem A. *For any fixed $r \geq 4$, there is a constant $c_r > 2^{-2^{6r}}$ satisfying the following condition.*

Every n-element point set P in general position in the plane has r pairwise disjoint subsets P_i $(1 \leq i \leq r)$ such that $|P_i| \geq \lfloor c_r n \rfloor$ and no matter how we pick one point from each P_i, they are in convex position.

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This result provides a *canonical* way to find many convex r-gons in a sufficiently large point set in the plane.

Bisztriczky and Fejes Tóth [BF] found the following generalization of the Erdős-Szekeres theorem to families of pairwise disjoint compact convex sets in the plane. We say that such a family $\mathcal{F}$ is in *general position* if none of its members is contained in the convex hull of the union of two others. $\mathcal{F}$ is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others.

Theorem B. *For every $r \geq 3$, there exists an integer $g(r)$ such that any family of at least $g(r)$ pairwise disjoint compact convex sets in general position in the plane has r members in convex position.*

In Section 3 of this note, we apply a straightforward counting argument suggested in [S] to establish the following common generalization of Theorems A and B.

Theorem 1. *For every $r \geq 4$, there is a positive constant $c_r = 2^{-O(r^2)}$ with the following property.*

Every family $\mathcal{F}$ of n pairwise disjoint compact convex sets in general position in the plane has r disjoint $\lfloor c_r n \rfloor$-membered subfamilies $\mathcal{F}_i$ $(1 \leq i \leq r)$ such that no matter how we pick one set from each $\mathcal{F}_i$, they are always in convex position.

It is worth mentioning that in the special case when all members of $\mathcal{F}$ are single points, our proof shows that the statement of Theorem A is true with a much better constant c_r than the one given in [BV].

The proofs of the next three theorems follow the same scheme.

A polygonal path $p_1 p_2 \ldots p_r$ in the plane or in space, is called *ε-straight* if $\angle p_{i-1} p_i p_{i+1} > \pi - \varepsilon$, $1 < i < r$ (cf. [ES2], [P]). The *length* of a polygonal path is the number of its *vertices*.

Theorem 2. *For every $d \geq 2, r \geq 3$ and $\varepsilon > 0$, there exists a positive constant $c = c^d_{r,\varepsilon}$ with the following property.*

Every n-element point set P in general position in Euclidean d-space has r pairwise disjoint subsets P_i $(1 \leq i \leq r)$ with at least $\lfloor cn \rfloor$ elements such that no matter how we pick a point from each P_i, they always form an ε-straight polygonal path.

Theorem 3. *For every $r, s \geq 2$, there exists a positive constant $c = c_{r,s} = (rs)^{-O(r)}$ with the following property.*

Let $\mathcal{F}$ be a family of n compact convex sets in the plane, no s of which are pairwise intersecting. Then $\mathcal{F}$ has r disjoint $\lfloor cn \rfloor$-membered subfamilies $\mathcal{F}_i$ $(1 \leq i \leq r)$ such that no two sets belonging to distinct subfamilies have a point in common.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any positive integer r, let $G(r)$ denote the graph obtained from G by replacing each vertex $v \in V(G)$ by r vertices, v_i $(1 \leq i \leq r)$, and connecting v_i and u_j by an edge if and only if $vu \in E(G)$ $(1 \leq i, j \leq r)$.

Theorem 4. *For every $c > 0, r \geq 1$, there exists a constant $c_r > 0$ with the following property.*

Let T be any tree of at most $c_r n$ vertices. Then every graph G with n vertices and at least cn^2 edges has a subgraph isomorphic to $T(r)$.

The letters $c, c_r, c_{r,\varepsilon}$, etc. appearing in different theorems denote unrelated positive constants depending on r, ε, etc.

2 Proofs of Theorems 2-3

To establish Theorem 2, we need the following straightforward generalization of a result from [ES2].

Lemma 2.1 *There exists a constant $c > 0$ such that any set of at least $k^{(c/\varepsilon)^{d-1}}$ points in Euclidean d-space has k elements that form an ε-straight polygonal path of length k.*

Proof of Theorem 2. Let ε, d, and r be fixed, and set $k = 2r - 1$. By Lemma 2.1, there exists an integer $K = K(\varepsilon, d, r)$ such that any set of K points in d-space contains k elements that form the vertex set of an $\varepsilon/3$-straight polygonal path Π. Notice that if we skip every other vertex of Π, then we obtain a polygonal path Π' with r vertices, which is ε-straight. The sequence formed by the $r - 1$ vertices we skipped is called the *support* of Π.

Consider now any set P of n points in the plane. Clearly, P contains at least

$$\binom{n}{K} \Big/ \binom{n-k}{K-k} = \binom{n}{k} \Big/ \binom{K}{k}$$

different $\varepsilon/3$-straight polygonal paths of length k, and at least

$$\frac{\binom{n}{k} / \binom{K}{k}}{n!/(n-r+1)!} > \frac{n^r}{K^{2r-1}}$$

of them must share the same support S.

Let P_i denote the set of all elements of P that occur as the $(2i-1)$-st vertex in some $\varepsilon/3$-straight polygonal path of length k, whose support is S $(1 \leq i \leq r)$. These sets meet the requirements of the theorem. In particular, for every i, we have

$$|P_i| > \frac{n^r}{K^{2r-1}} \Big/ \prod_{j \neq i} |P_j| > \frac{n}{K^{2r-1}}. \quad \square$$

The proof of Theorem 3 uses the same idea. We need a little preparation.

Let $\mathcal{F}$ be a family of n compact convex sets in the plane. Assume without loss of generality that no two members of $\mathcal{F}$ have a common vertical tangent line. For $C \in \mathcal{F}$, let $\pi(C)$ denote the projection of C onto the x-axis. Following [LMPT], we define four partial orders, $\prec_1$, $\prec_2$, $\prec_3$ and $\prec_4$, on $\mathcal{F}$. For any two disjoint sets $A, B \in \mathcal{F}$,

1. $A \prec_1 B$ if $\pi(A) \subseteq \pi(B)$ and A lies below B ("below" means in the y-axis direction).
2. $A \prec_2 B$ if $\pi(A) \subseteq \pi(B)$ and A lies above B.
3. $A \prec_3 B$ if the left endpoint of $\pi(B)$ is to the right of the left endpoint of $\pi(A)$, the right endpoint of $\pi(B)$ is to the right of the right endpoint of $\pi(A)$ and in the part where $\pi(A)$ and $\pi(B)$ overlap (if any), A lies above B.
4. $A \prec_4 B$ if the left endpoint of $\pi(B)$ is to the right of the left endpoint of $\pi(A)$, the right endpoint of $\pi(B)$ is to the right of the right endpoint of $\pi(A)$ and in the part where $\pi(A)$ and $\pi(B)$ overlap (if any), A lies below B.

Lemma 2.2 [LMPT]. *Any family of more than $(k-1)^4(s-1)$ compact convex sets in the plane, no s of which have pairwise nonempty intersections, contains k members that form a chain with respect to one the relations $\prec_1, \prec_2, \prec_3, \prec_4$.*

Proof of Theorem 3. Setting $K = (k-1)^4(s-1)+1$ and $k = 2r-1$, we obtain just like in the previous proof that there exists $1 \le j \le 4$ such that $\mathcal{F}$ has at least $\frac{1}{4}\binom{n}{k}/\binom{K}{k}$ chains $\mathcal{C}$ of length k with respect to $\prec_j$. If we skip every other element of $\mathcal{C}$, we obtain a chain $\mathcal{C}'$ of length r. The chain $\mathcal{C} \setminus \mathcal{C}'$ is called the *support* of $\mathcal{C}$. It follows that at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{\binom{n}{r-1}} > \frac{n^r}{K^{2r-1}}$$

chains C share the same support S.

For every i $(1 \le i \le r)$, let $\mathcal{F}_i$ denote the set of all members of $\mathcal{F}$ that occur as the $(2i-1)$-st smallest element of a chain in $(\mathcal{F}, \prec_j)$ with length k and support S. It is clear that no two sets belonging to distinct $\mathcal{F}_i$'s have a point in common. The same estimation as at the end of the proof of Theorem 2 gives that $|\mathcal{F}_i| \ge \frac{n}{K^{2r-1}}$ for every i. □

It is possible that the following far-reaching generalization of Theorem 3 is also true. For every $s \ge 2$, there exists a constant $c = c_s > 0$ with the property that any family of n compact connected sets in the plane, no s of which have pairwise nonempty intersections, has at least cn pairwise disjoint members. We have been unable to decide whether this statement holds for families of straight-line segments.

3 Proof of Theorem 1

We follow the same approach as in the previous section. The proof is based on a stronger version of Theorem B.

Lemma 3.1 [PT]. *For every $k \geq 3$, any family of 2^{4k} pairwise disjoint compact convex sets in general position in the plane has k members in convex position.*

Let $\mathcal{F}$ be a family of n pairwise disjoint compact convex sets in general position in the plane. Assume without loss of generality that no three members of $\mathcal{F}$ have a common tangent line and no two have a common vertical tangent.

Applying first Lemma 2.2 and then Lemma 3.1, we obtain that for every k, any 2^{16k}-membered subfamily of $\mathcal{F}$ contains k sets in convex position that form a chain with respect to one of the relations $\prec_j$ $(1 \leq j \leq 4)$.

Set $k = 4r-2, K = 2^{16k}$. Just like in the proof of Theorem 3, it follows that there exists $1 \leq j \leq 4$ such that $\mathcal{F}$ has at least $\frac{1}{4}\binom{n}{k}/\binom{K}{k}$ chains $\mathcal{B} = (B_1 \prec_j B_2 \prec_j \ldots \prec_j B_k)$, whose members are in convex position. We distinguish two substantially different cases according to the value of j.

Case 1: $j = 1$. Let $\mathcal{B}$ be any chain of length $k = 4r-2$ with respect to $\prec_1$, whose members are in convex position. Then $\mathcal{B}$ has a subchain $\mathcal{C} = (C_1, C_2, \ldots, C_{2r-1})$ with the following property. Each C_i contributes to the boundary of the convex hull conv $\cup_{i=1}^{2r-1} C_i$ at least one point to the left of C_1, or each C_i contributes to bd conv $\cup_{i=1}^{2r-1} C_i$ at least one point to the right of C_1. In the former case we call $\mathcal{C}$ a *left-convex* chain and in the latter one a *right-convex* chain. Thus, there are at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{2\binom{n-2r+1}{2r-1}}$$

different chains $\mathcal{C} = (C_1 \prec_1 C_2 \prec_1 \ldots \prec_1 C_{2r-1})$ of the same type, say, left-convex. Define the *support* of $\mathcal{C}$ as the subchain $\mathcal{C}^* \subseteq \mathcal{C}$ formed by the even-numbered elements, i.e., let

$$\mathcal{C}^* = (C_2, C_4, C_6, \ldots, C_{2r-2}).$$

Clearly, there are at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{2\binom{n-2r+1}{2r-1}\binom{n}{r-1}} > \frac{n^r}{K^{4r-2}}$$

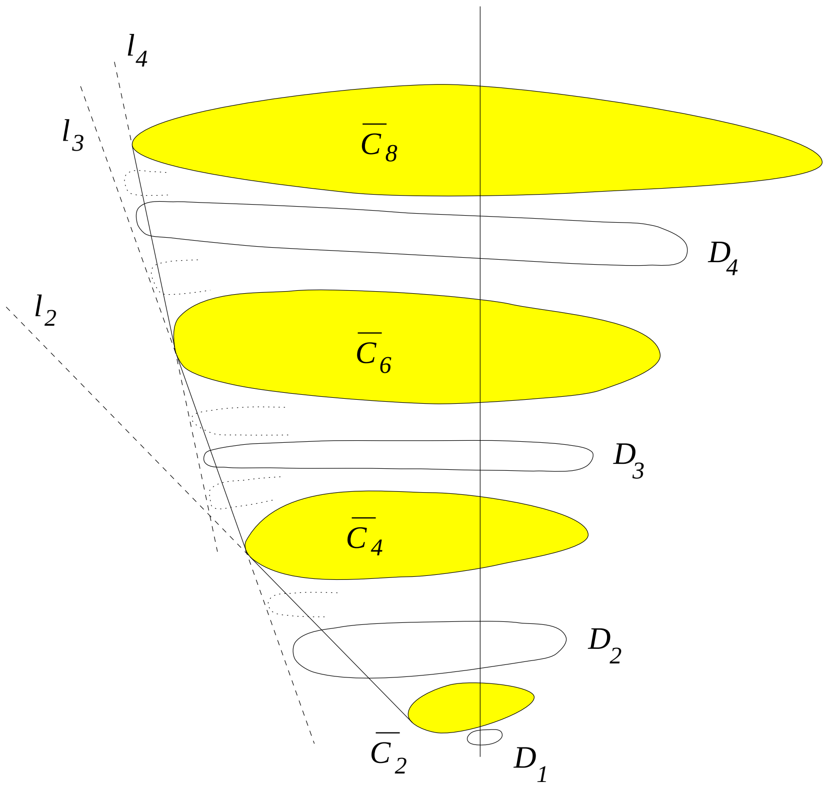
different left-convex chains $\mathcal{C} = (C_1, C_2, \ldots, C_{2r-1})$ which have the same support. These chains are called *standard*. Let

$$(\bar{C}_2, \bar{C}_4, \bar{C}_6, \ldots, \bar{C}_{2r-2})$$

denote the common support of the standard chains. We will refer to this sequence as the *standard support.*

For any t $(1 \leq t \leq r)$, let $\mathcal{F}_t$ denote the family of all members of $\mathcal{F}$ that occur as the $(2t-1)$-st element $C_{2t-1} \in \mathcal{C}$ for some standard chain $\mathcal{C}$. We have

$$|\mathcal{F}_t| > \frac{n^r}{K^{4r-2}} \Big/ \prod_{s \neq t} |\mathcal{F}_s| > \frac{n}{K^{4r-2}}.$$



Figure

It remains to show that for every choice $D_t \in \mathcal{F}_t$ $(1 \leq t \leq r)$, the sets $D_1, D_2, \ldots, D_r$ are in convex position (cf. Figure). To see this, consider the left-hand side ∂ of the boundary of the union of all members of the standard support. ∂ consists of nonempty portions of the boundaries of the sets $\bar{C}_{2t}$ $(1 \leq t < r)$, separated by straight-line segments. For any $1 < t < r$, let l_t denote the the common tangent line of the sets $\bar{C}_{2t-2}$ and $\bar{C}_{2t}$ with the property that every other member of the standard support is on its right-hand side. To finish the proof in Case 1, it is sufficient to notice that every $D_t \in \mathcal{F}_t$ has at least one point to the left of l_t, while all members of $\cup_{s\neq t}\mathcal{F}_s$ lie on the right-hand side of l_t. Therefore, $D_1, D_2, \ldots, D_r$ are in convex position.

Case 2: $j = 3$. Let $\mathcal{B}$ be any chain of length $k = 4r - 2$ with respect to $\prec_3$, whose members are in convex position. Then $\mathcal{B}$ has a subchain $\mathcal{C} = (C_1, C_2, \ldots, C_{2r-1})$ with the following property. Each C_i contributes at least one point to the upper portion of $\operatorname{bd}\operatorname{conv} \cup_{i=1}^{2r-1} C_i$, or each C_i contributes at least one point to the lower portion of $\operatorname{bd}\operatorname{conv} \cup_{i=1}^{2r-1} C_i$. In the former case, $\mathcal{C}$ is called a *upper-convex* chain, and in the latter one, a *lower-convex* chain. Thus, there are at least

$$\frac{\frac{1}{4}\binom{n}{k}/\binom{K}{k}}{2\binom{n-2r+1}{2r-1}}$$

different chains $\mathcal{C} = (C_1 \prec_3 C_2 \prec_3 \ldots \prec_3 C_{2r-1})$ of the same type, say, upper-convex. The rest of the argument is exactly the same as in Case 1, with the only difference that in place of the left-hand side ∂ of the boundary of the union of all members of the standard support, we have to consider its *upper side*.

The cases $j = 2$ and $j = 4$ are symmetric counterparts of the above two cases, so they do not have to be treated separately.

4 Proof of Theorem 4

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Assume that $|V(G)| = n$ and $|E(G)| \geq cn^2$ for some constant $c > 0$, and let r be a fixed positive integer.

First, we would like to show that G contains many complete bipartite subgraphs $K_{r,r}$ with r vertices in its classes. The proof is based on the following simple statement, discovered by Erdős, which is a weak version of a result of [KST].

Lemma 4.1 *For every $r \geq 1$ and every $\gamma > 0$, there exists a positive integer $n_0 = n_0(r, \gamma)$ with the following property.*

Every graph G_0 with n_0 vertices and at least γn_0^2 edges contains a complete bipartite subgraph $K_{r,r}$ with r vertices in each of its classes.

Let x denote the number of n_0-element subsets of $V(G)$ which induce a subgraph of G with at least γn_0^2 edges. Then we have

$$x\binom{n_0}{2} + \left(\binom{n}{n_0} - x\right)\gamma n_0^2 > |E(G)|\binom{n-2}{n_0-2} \geq cn^2\binom{n-2}{n_0-2},$$

which yields

$$x > \binom{n}{n_0}\frac{c(n_0-1) - \gamma n_0}{\frac{n_0-1}{2} - \gamma n_0}.$$

Thus, for $\gamma = c/2, n_0 > 2$, we obtain $x > \frac{c}{2}\binom{n}{n_0}$.

Set $n_0 = n_0(r, \gamma) = n_0(r, c/2)$. By Lemma 4.1, every subgraph of G with n_0 vertices and at least $(c/2)n_0^2$ edges contains at least one copy of $K_{r,r}$. Thus, the number y of complete bipartite subgraphs $K_{r,r}$ of G satisfies

$$y \geq \frac{x}{\binom{n-2r}{n_0-2r}} > \frac{\frac{c}{2}\binom{n}{n_0}}{\binom{n-2r}{n_0-2r}} > \frac{cn^{2r}}{2n_0^{2r}}.$$

Suppose for simplicity that n is divisible by r, and consider all possible partitions of $V(G)$ into classes of size r. The number of these partitions is

$$p(n, r) = \frac{\binom{n}{r}\binom{n-r}{r}\binom{n-2r}{r}\cdots\binom{r}{r}}{(n/r)!}.$$

For every partition P, construct a graph $G(P)$ whose vertices are the classes V_i $(1 \leq i \leq n/r)$ of the partition, and two vertices V_i and V_j are connected by an edge of $G(P)$ if and only if G contains all edges running between them. By averaging over all partitions, we find that there exists a P such that the number of edges of $G(P)$ is at least

$$\frac{yp(n-2r, r)}{p(n, r)} > \frac{cn^{2r}}{2n_0^{2r}} \frac{\binom{\frac{n}{r}}{1}\binom{\frac{n-r}{r}}{1}}{\binom{n}{r}\binom{n-r}{r}} > c\left(\frac{r}{e}\right)^{2r} \cdot \left(\frac{n}{r}\right)^2.$$

We can now finish the proof of the theorem by applying to $G(P)$ the following simple assertion, whose proof is left to the reader.

Lemma 4.2 *For any $C > 0$, every graph with N vertices and at least CN^2 edges contains every tree of at most $CN/2$ vertices as a subgraph.*

Hence, Theorem 4 is true with $c_r = \frac{c}{2}\left(\frac{r}{e}\right)^{2r}$.

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