TWO EXTENSIONS OF THE ERDŐS–SZEKERES PROBLEM

ANDREAS F. HOLMSEN[†], HOSSEIN NASSAJIAN MOJARRAD[◇],
JÁNOS PACH[◇], AND GÁBOR TARDOS[‡]

ABSTRACT. According to Suk's breakthrough result on the Erdős–Szekeres problem, any point set in general position in the plane, which has no n elements that form the vertex set of a convex n-gon, has at most $2^{n+O(n^{2/3} \log n)}$ points. We strengthen this theorem in two ways. First, we show that the result generalizes to convexity structures induced by pseudoline arrangements. Second, we improve the error term.

A family of n convex bodies in the plane is said to be in *convex position* if the convex hull of the union of no $n - 1$ of its members contains the remaining one. If any *three* members are in convex position, we say that the family is in *general position*. Combining our results with a theorem of Dobbins, Holmsen, and Hubard, we significantly improve the best known upper bounds on the following two functions, introduced by Bisztriczky and Fejes Tóth and by Pach and Tóth, respectively. Let $c(n)$ (and $c'(n)$) denote the smallest positive integer N with the property that any family of N pairwise disjoint convex bodies in general position (resp., N convex bodies in general position, any pair of which share at most two boundary points) has an n-membered subfamily in convex position. We show that

$$c(n) \le c'(n) \le 2^{n+O\left(\sqrt{n \log n}\right)}.$$

1. INTRODUCTION

We say that a set of n points in the plane is in *convex position* if the convex hull of no $n - 1$ of them contains the n-th point. If no three elements of the set are collinear (that is, any three points are in convex position), then the set is said to be in *general position*. According to a classical conjecture of Erdős and Szekeres [7], if P is a set of points in general position in the plane with $|P| \ge 2^{n-2} + 1$, then it has n elements in convex position. This bound, if true, cannot be improved [8]. In a recent breakthrough, Andrew Suk [19] came close to proving the conjectured bound.

Theorem 1.1 (Suk, 2017). *Given any integer $n \ge 3$, let $e(n)$ denote the smallest number with the property that every family of at least $e(n)$ points in general position in the plane has n elements in convex position. Then we have*

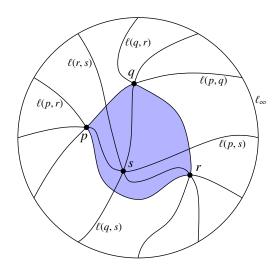$$e(n) \le 2^{n+O(n^{2/3} \log n)}.$$

FIGURE 1. A pseudo-configuration of four points with the convex hull shaded.

A set of simple continuous curves in the Euclidean plane that start and end "at infinity" is called an *arrangement of pseudolines* if any two of them meet in precisely one point: at a proper crossing. A *pseudo-configuration* is a finite set of points P in the Euclidean plane such that each pair of distinct points p and q in P span a unique *pseudoline*, denoted by $\ell(p, q)$ such that $L(P) = \{\ell(p, q) : p, q \in P, p \neq q\}$ form a pseudoline arrangement and for any $p, q \in P$, $p \neq q$ we have $\ell(p, q) = \ell(q, p)$ and $\ell(p, q) \cap P = \{p, q\}$; see [10].

This underlying pseudoline arrangement induces a convexity structure on the point configuration in a natural way. For any pair of points $p, q \in P$, the bounded portion of $\ell(p, q)$ between p and q is called the *pseudosegment* connecting p and q. If we delete from the plane all pseudosegments between the elements of P, the plane is divided into a number of connected components, precisely one of which is unbounded. The *convex hull* of the configuration is defined as the complement of the unbounded region, and is denoted by conv$P$. We say that a subset $Q \subseteq P$ is in *convex position* if no point $p \in Q$ is in the convex hull of $Q \setminus \{p\}$.[1]

It turns out that for four points there are only two combinatorially distinct pseudo-configurations and both can be obtained from straight lines, but for five or more points there exist pseudo-configurations that are not realizable by straight lines. Still, the number of possible pseudo-configurations on five points is limited and we will leave the verification of some simple statements about at most five points in a pseudo-configuration to the reader.

Many basic theorems of convexity hold in this more general setting. For instance, a set of points is in convex position if and only if every four of its elements are in convex position [5]. This is Carathéodory's theorem in the plane.

---

[1]Pseudo-configurations also have a purely combinatorial characterization. They can be defined by several equivalent systems of axioms. Other names for pseudo-configurations that can be found in the literature are *generalized configurations* [10], *uniform rank 3 acyclic oriented matroids* [4], and *CC-systems* [13].

Goodman and Pollack [11] proposed the generalization of the Erdős-Szekeres problem to pseudo-configurations. The original "cup-cap" proof due to Erdős and Szekeres [7] readily generalizes to this setting:

Theorem 1.2. *Let P be a pseudo-configuration. If $|P| \geq 4^k$, then P contains a k-element subset in convex position.*

The purpose of this note is to show that Suk's breakthrough result, Theorem 1.1 carries over to pseudo-configurations. In the process we also improve on the error term.

Theorem 1.3. *Given any $n \geq 3$, let $b(n)$ denote the smallest number such that every pseudo-configuration of size at least $b(n)$ has n members in convex position. Then we have*
$$b(n) \leq 2^{n+O\left(\sqrt{n \log n}\right)}.$$

Clearly, $b(n) \geq e(n)$ holds for all n, thus our results also bounds the function $e(n)$ defined for the original Erdős-Szekeres problem (cf. Theorem 1.1).

Bisztriczky and G. Fejes Tóth [2, 3] gave another (seemingly unrelated) generalization of the Erdős-Szekeres problem in 1989 by replacing point sets with families of pairwise disjoint convex bodies. They defined n convex bodies to be in *convex position* if the convex hull of no $n - 1$ of them contains the remaining one. If any three members of a family of convex bodies are in convex position, then the family is in *general position*. In their pioneering paper, Bisztriczky and Fejes Tóth proved that for any $n \geq 3$, there exists a smallest integer $c(n)$ with the following property. If $\mathcal{F}$ is a family of pairwise disjoint convex bodies in general position in the plane with $|\mathcal{F}| \geq c(n)$, then it has n members in convex position. They conjectured that $c(n) = e(n)$. The first singly-exponential upper bound on $c(n)$ was established by Pach and Tóth [16]. They extended the statement to families of pairwise *noncrossing* convex bodies, that is, to convex bodies that may intersect, but any pair can share at most two boundary points [17]. This assumption is necessary.

Theorem 1.4 (Pach–Tóth, 2000). *For any integer $n \geq 3$, there exists a smallest number $c'(n)$ with the following property. Any family of at least $c'(n)$ pairwise noncrossing convex bodies in general position in the plane has n members in convex position.*

Clearly, we have $c'(n) \geq c(n) \geq e(n)$ for every n. The original upper bound on $c'(n)$ was subsequently improved by Hubard, Montejano, Mora, and Suk [12] and by Fox, Pach, Sudakov, and Suk [9] to $2^{O(n^2 \log n)}$, and later by Dobbins, Holmsen, and Hubard [6] to 4^n. More importantly from our point of view, they showed that there is an intimate relationship between the generalizations of the Erdős-Szekeres problem to non-crossing convex bodies and to pseudo-configurations. The following is the union of Lemmas 2.4 and 2.7 in their paper.

Theorem 1.5 (Dobbins–Holmsen–Hubard, 2014). *Let $\mathcal{F}$ be a family of pairwise noncrossing convex bodies in general position in the plane. There exists a pseudo-configuration P and a bijection $\varphi : P \to \mathcal{F}$ such that for any subset $S \subseteq P$ which is in convex position, the subfamily $\varphi(S)$ is also in convex position.*

It follows from this result that $c'(n) \leq b(n)$ for all n. (In fact, it was shown in [6] that $c'(n) = b(n)$ for all $n \geq 3$.) In view of this, Theorem 1.3 immediately implies the following.

Theorem 1.6. *Given any $n \geq 3$, let $c'(n)$ denote the smallest number such that every family of at least $c'(n)$ pairwise noncrossing convex bodies in general position in the plane has n members in convex position. Then we have*

$$c'(n) \leq 2^{n+O\left(\sqrt{n \log n}\right)}.$$

The rest of this note is organized as follows. After highlighting two auxiliary results in Section 2, we present the proof of Theorem 1.3 in Section 3.

2. Auxiliary results

To follow Suk's line of argument, we recall two results needed for the proof: a combinatorial version of the "cup-cap" theorem (Theorem 2.1) and a variant of a positive fraction Erdős–Szekeres theorem [1] (Theorem 2.4). For future reference, we also collect some simple observations on pseudo-configurations in convex positions (Observations 2.2 and 2.3).

Transitive colorings. Let S be a finite set with a given linear ordering $\prec$, and suppose the ordered triples $s_i \prec s_j \prec s_k$ are partitioned into two parts $T_1 \cup T_2$. This partition is called a *transitive coloring* if every $s_1 \prec s_2 \prec s_3 \prec s_4$ in S and $i \in \{1, 2\}$ satisfy

$$(s_1, s_2, s_3), (s_2, s_3, s_4) \in T_i \Rightarrow (s_1, s_2, s_4), (s_1, s_3, s_4) \in T_i.$$

Transitive colorings were introduced in [9] and [12]. The following statement can be proved in precisely the same way as the "cup-cap" theorem; see [14] for an alternative proof.

Theorem 2.1. [9, 12] *Let S be a finite set with a given linear ordering and let $T_1 \cup T_2$ be a transitive coloring of the triples of S. If*

(1) $$|S| > \binom{k + l - 4}{k - 2},$$

then there exists a k-element subset $S_1 \subseteq S$ such that every triple of S_1 is in T_1, or an l-element subset $S_2 \subseteq S$ such that every triple of S_2 is in T_2.

Convex hulls of pseudo-configurations. Below we collect a few simple observations on the convexity structure of pseudo-configurations. These statements are trivial for the usual notion of convexity, and easy to prove in this more general context.

Observation 2.2. *Let P be a pseudo-configuration.*
   (i) *The convex hull is a monotone operation. That is, for any $X \subseteq Y \subseteq P$, we have $\mathrm{conv}X \subseteq \mathrm{conv}Y$.*
  (ii) *$\mathrm{conv}X$ is a simply connected closed set, for any $X \subseteq P$.*
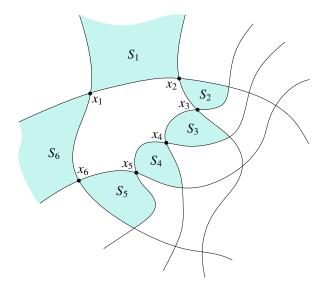 (iii) *If $X \subseteq P$ is in convex position, then all points of X appear on the boundary of $\mathrm{conv}X$.*

FIGURE 2. A pseudo-configuration in convex position where the spikes are shaded.

(iv) *Let $k \geq 3$, and assume that $X = \{x_1, x_2, \ldots, x_k\} \subseteq P$ is in convex position, where the points x_i appear on the boundary of $\mathrm{conv}X$ in this cyclic order. Then the boundary of $\mathrm{conv}X$ is the union of the pseudosegments $\mathrm{conv}\{x_i, x_{i+1}\}$ for $1 \leq i \leq k$. Furthermore, for each i, the pseudoline $\ell(x_i, x_{i+1})$ intersects $\mathrm{conv}X$ in the pseudosegment $\mathrm{conv}\{x_i, x_{i+1}\}$, and (the rest of) $\mathrm{conv}X$ lies entirely on one side of $\ell(x_i, x_{i+1})$. (Indices are understood modulo k.)* □

Convex clusterings. Consider the pseudo-configuration described in Observation 2.2(iv): Let $X = \{x_1, x_2, \ldots, x_k\}$ be a k-element subset of P in convex position, where $k \geq 3$, and suppose that the points x_i appear on the boundary of $\mathrm{conv}X$ in this cyclic order. We define the *i*-th *spike* of X, denoted by S_i, to be the open region consisting of the points of the plane separated from the interior of $\mathrm{conv}X$ by the pseudoline $\ell(x_i, x_{i+1})$, but not separated from $\mathrm{conv}X$ by $\ell(x_{i-1}, x_i)$ and by $\ell(x_{i+1}, x_{i+2})$. This is a connected region bounded by the pseudosegment $\mathrm{conv}\{x_i, x_{i+1}\}$ and by portions of the pseudolines $\ell(x_{i-1}, x_i)$ and $\ell(x_{i+1}, x_{i+2})$. It is either a triangle-like bounded region or an unbounded region of three sides; see Fig. 2.

Observation 2.3. *Let $1 \leq i \leq k$.*

(i) *The line $\ell(x_i, x_{i+1})$ is disjoint from every spike and separates S_i from all other spikes S_j ($j \neq i$). In particular, the spikes are pairwise disjoint.*

(ii) *A point $p \in P \setminus X$ belongs to the spike S_i if and only if $X' = X \cup \{p\}$ is in convex position and p appears on the boundary of $\mathrm{conv}X'$ between x_i and x_{i+1}. In particular, whether $X \cup \{p\}$ is in convex position is determined by which region p belongs in the arrangement of pseudolines spanned by X.* □

For the usual notion of convexity in the Euclidean plane, the following statement was proved by Pór and Valtr [18]. It is a slight strengthening of a result of Pach

and Solymosi [15] that can be obtained by simple double counting. Since we will use this statement for pseudo-configurations, to make our paper self-contained, we translate its proof into this setting.

Theorem 2.4. *Let $k \geq 3$ be an integer, and let P be a pseudo-configuration with $|P| = N \geq 2^{4k}$. Then there exists a subset $X = \{x_1, x_2, \ldots, x_k\} \subset P$ in convex position such that the sets P_i of all points of P lying in the i-th spike, $i = 1, \ldots, k$, satisfy the inequality*

$$(2) \qquad \prod_{i=1}^{k} |P_i| \geq \frac{N^k}{2^{8k^2}}.$$

Proof. Let P be a pseudo-configuration with $|P| = N \geq 2^{4k}$. By Theorem 1.2, every 4^{2k}-element subset $Q \subseteq P$ contains a $2k$-element subset $R \subset Q$ in convex position. Therefore, by double-counting, P has at least

$$\frac{\binom{N}{4^{2k}}}{\binom{N-2k}{4^{2k}-2k}} = \frac{\binom{N}{2k}}{\binom{4^{2k}}{2k}} > \left(\frac{N}{4^{2k}}\right)^{2k}$$

distinct $2k$-element subsets in convex position.

Given a $2k$-element subset Y in convex position, we say that a k-element subset $X \subset Y$ *supports* Y if the points of Y along the boundary of conv$Y$ alternately belong to X and $Y \setminus X$. Note that Y is supported by two subsets.

Since the number of k-element subsets of P in convex position is at most $\binom{N}{k}$, there exists a k-element subset X which supports at least

$$\frac{\left(\frac{N}{4^{2k}}\right)^{2k}}{\binom{N}{k}} > \frac{N^k}{2^{8k^2}}$$

distinct $2k$-element subsets in convex position. By Observation 2.3(ii), if X supports Y, then the points of $Y \setminus X$ belong to distinct spikes of X, which implies inequality (2). $\qquad\square$

3. Proof of Theorem 1.3

Consider a sufficiently large fixed pseudo-configuration P, let $k \geq 4$ be an even integer, and let $X = \{x_1, x_2, \ldots, x_k\} \subset P$ be a k-element subset in convex position such that its points appear on the boundary of conv$X$ in this cyclic order. Suppose that X meets the requirements of Theorem 2.4. As before, let $S_1, S_2, \ldots, S_k$ denote the spikes of X and let $P_i = P \cap S_i$. The indices are taken modulo k.

Vertical and horizontal orderings on P_i. Let p and q be distinct points in P_i. We write

$$p <_i^v q \iff \text{conv}\{x_{i-1}, p, x_{i+2}\} \subset \text{conv}\{x_{i-1}, q, x_{i+2}\},$$
$$p <_i^h q \iff \text{conv}\{x_{i-1}, q\} \cap \text{conv}\{x_{i+2}, p\} \neq \emptyset,$$

where the superscripts v and h refer to the adjectives "vertical" and "horizontal", respectively.

Observation 3.1. *Let $1 \leq i \leq k$.*

(i) *Both $\prec_i^v$ and $\prec_i^h$ are partial orders on P_i.*

(ii) *Any two distinct elements of P_i are comparable by either $\prec_i^v$ or $\prec_i^h$, but not by both.*

Proof. The definition of $\prec_i^v$ clearly implies that it is a partial order. To see that the same is true for $\prec_i^h$, one has to show that if $p \prec_i^h q \prec_i^h r$ for three points $p, q, r \in P_i$, then $p \prec_i^h r$. This can be done by checking the few possible pseudo-configurations of the five points x_{i-1}, x_{i+2}, p, q and r.

To prove (ii), it is sufficient to consider the pseudo-configurations consisting of only four points: x_{i-1}, x_{i+2}, and two points p and q from P_i. Using the fact that p and q lie on the same side of $\ell(x_{i-1}x_{i+2})$, one can show that out of the four relations $p \prec_i^v q$, $q \prec_i^v p$, $p \prec_i^h q$, and $q \prec_i^h p$, precisely one will hold. Consider the four open regions into which the pseudolines $\ell(px_{i-1})$ and $\ell(px_{i+2})$ partition the plane. The region in which q lies uniquely determines which of the above four relations will hold. □

For $1 \le i \le k$, let v_i denote the length of the longest chain in P_i with respect to $\prec_i^v$, and let h_i denote the length of the longest chain in P_i with respect to $\prec_i^h$. By Observation 3.1 and by (the easy part of) Dilworth's theorem, we have

$$(3) \qquad\qquad |P_i| \le v_i h_i.$$

Further observations concerning points and spikes. As before, the following observations are trivial for the usual notion of convexity in the Euclidean plane. Here we show that they also hold for pseudo-configurations.

Observation 3.2. *For any pair of distinct points $p, q \in P$, the pseudoline $\ell(p, q)$ intersects at most two spikes of X.*

Proof. Assume for contradiction that $\ell(p, q)$ intersects three separate spikes S_i, S_j, and S_l in this order. By Observation 2.3(i), this line should intersect $\ell(x_j, x_{j+1})$ twice, a contradiction. □

Observation 3.3. *Let p and q be distinct points of P_i. If $p \prec_i^v q$, then the pseudoline $\ell(p, q)$ separates spikes S_{i-1} and S_{i+1}.*

Proof. Since $p \in \text{conv}\{x_{i-1}, q, x_{i+2}\}$, the pseudoline $\ell(p, q)$ intersects the pseudosegment $\text{conv}\{x_{i-1}, x_{i+2}\}$. This implies that $\ell(p, q)$ has to intersect one of the spikes $S_{i+2}, S_{i+3}, \ldots, S_{i-2}$. By Observation 3.2, $\ell(p, q)$ intersects at most two spikes, one of which is S_i. Thus, it cannot intersect S_{i-1} and S_{i+1}, which implies that S_{i-1} and S_{i+1} must be separated by $\ell(p, q)$. □

Observation 3.4. *Let p and q be distinct points of P_i. If $p \prec_i^h q$, then all the spikes $S_{i+2}, S_{i+3}, \ldots, S_{i-2}$ must lie on the same side of the pseudoline $\ell(p, q)$.*

Proof. All spikes S_j with $j \notin \{i-1, i+1\}$ are on the same side of both pseudolines $\ell(x_{i-1}, x_i)$ and $\ell(x_{i+1}, x_{i+2})$. The angular region determined by these two pseudolines and containing the above spikes (and the interior of $\text{conv}X$) is cut into two parts by the pseudosegment $\text{conv}\{x_{i-1}, x_{i+2}\}$, so that S_i lies on one side and the spikes S_j with $j \notin \{i-1, i, i+1\}$ on the other. Our assumption $p \prec_i^h q$ implies that the pseudoline $\ell(p, q)$ does not intersect the pseudosegment $\text{conv}\{x_{i-1}, x_{i+2}\}$, so the part
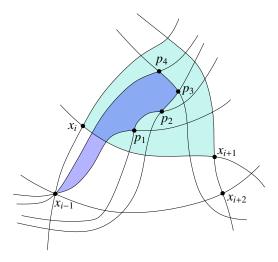
FIGURE 3. A left convex chain $p_1 \prec_i^v p_2 \prec_i^v p_3 \prec_i^v p_4$ in P_i with conv$\{p_1, p_2, p_3, p_4, x_{i-1}\}$ in darker shade.

of the angular region on the other side of this pseudosegment (including all relevant spikes) is on the same side of $\ell(p, q)$, as claimed.   □

Vertical convex chains. Let $C \subseteq P_i$ be a chain with respect to $\prec_i^v$. If $\{x_{i-1}\} \cup C$ is in convex position, we call C a *left convex chain* in P_i. If $\{x_{i+2}\} \cup C$ is in convex position, we call C a *right convex chain* in P_i.

Note that if $|C| = 3$, then C is either a left convex chain or a right convex chain, but not both. This can be verified by checking the pseudo-configuration $C \cup \{x_{i-1}, x_{i+2}\}$. Moreover, if we have $p_1 \prec_i^v p_2 \prec_i^v p_3 \prec_i^v p_4$ and both $\{p_1, p_2, p_3\}$ and $\{p_2, p_3, p_4\}$ are left (right) convex chains, then $\{p_1, p_2, p_3, p_4\}$ is also a left (right) convex chain. Therefore, the same holds for both $\{p_1, p_2, p_4\}$ and $\{p_1, p_3, p_4\}$. This can be verified by checking the pseudo-configuration $\{p_1, p_2, p_3, p_4, x_{i-1}, x_{i+2}\}$. See Fig. 3.

Take a chain $C \subseteq P_i$ of maximal size $|C| = v_i$, totally ordered by $\prec_i^v$. Partition the triples of C into left and right convex chains. In this way, we obtain a transitive coloring. Letting a_i and b_i denote the length of the longest left convex chain and the length of the longest right convex chain in C, respectively, by Theorem 2.1, we have

$$(4) \qquad\qquad v_i \leq \binom{a_i + b_i - 2}{a_i - 1}.$$

Actually, Theorem 2.1 only guarantees the existence of large subsets $C_1, C_2 \subseteq C$ such that all triples in C_1 are left convex chains and all triples in C_2 are right convex chains. However, using the above observations and the generalization of Carathéodory's theorem to pseudo-configurations, it follows that C_1 and C_2 themselves must form a left convex chain and a right convex chain, respectively.

Observation 3.5. *If R is a right convex chain in P_i and L is a left convex chain in P_{i+1}, then $R \cup L$ is in convex position.*
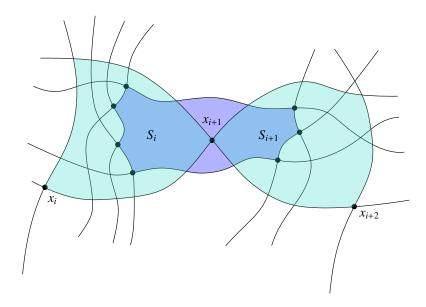
FIGURE 4. Joining a right convex chain $R \subseteq P_i$ and a left convex chain $L \subseteq P_{i+1}$ to form a subset in convex position (convex hull in darker shade).

Proof. First, note that for any pseudo-configuration P consisting of four points, if a point $p \in P$ lies in the convex hull of $P \setminus \{p\}$, then any pseudoline passing through p and any other point of P crosses the pseudosegment determined by the other two points of P.

To prove the observation, it is enough to show that any four points $p, q, r, s \in R \cup L$ are in convex position. If all of them lie in one of R or L, then we are clearly done. Assume first that $r, s \in R$ and $p, q \in L$. By Observation 3.3, the pseudolines $\ell(p, q)$ and $\ell(r, s)$ do not intersect the pseudosegments $\mathrm{conv}\{r, s\} \subset S_i$ and $\mathrm{conv}\{p, q\} \subset S_{i+1}$, respectively. Therefore, by the discussion above, the points p, q, r, s are in convex position.

Now consider the case where $p, q, r \in L$ and $s \in R$. Again by Observation 3.3, none of the pseudolines $\ell(p, q)$, $\ell(p, r)$, and $\ell(q, r)$ intersects the spike S_i. Therefore, x_i and s lie in the same open region determined by the arrangement of these three pseudolines. By the assumption, the set $\{p, q, r, x_i\}$ is in convex position, so by the last statement of Observation 2.3(ii) $\{p, q, r, s\}$ is in convex position, as well. The other case, $p \in L$ and $q, r, s \in R$, can be settled in a similar manner. See Fig. 4. $\square$

Horizontal convex chains. Let $C \subseteq P_i$ be a chain with respect to $\prec_i^h$. If $\{p, q, r, x_{i-1}, x_{i+2}\}$ is in convex position for any three distinct elements p, q, r of C, we call C an *inner convex chain*. If $\{p, q, r, x_{i-1}, x_{i+2}\}$ is *not* in convex position for any three distinct elements p, q, r of C, we call C an *outer convex chain*.

Note that chains of at most two elements are both inner and outer convex chains by this definition.
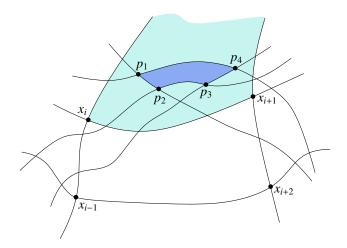
Observation 3.6. *Let $1 \le i \le k$.*

FIGURE 5. An outer convex chain $p_1 \prec_i^h p_2 \prec_i^h p_3 \prec_i^h p_4$ in P_i with the conv$\{p_1, p_2, p_3, p_4\}$ in darker shade.

(i) *The partitioning of the triples in a horizontal chain C (ordered by $\prec_i^h$) into inner and outer convex chains is a transitive coloring.*
(ii) *The inner and outer convex chains in P_i are in convex position.*

Proof. Consider a horizontal chain $p \prec_i^h q \prec_i^h r$ in P_i. By checking the pseudo-configuration $\{p, q, r, x_{i-1}, x_{i+2}\}$ we can verify that the following are equivalent:

- (p, q, r) is an outer (inner) convex chain.
- conv$\{x_{i-1}, x_{i+2}\}$ and r are separated by (lie on the same side of) $\ell(p, q)$.
- conv$\{x_{i-1}, x_{i+2}\}$ and p are separated by (lie on the same side of) $\ell(q, r)$.
- conv$\{x_{i-1}, x_{i+2}\}$ and q lie on the same side of (are separated by) $\ell(p, r)$.

Now consider a horizontal chain $p_1 \prec_i^h p_2 \prec_i^h p_3 \prec_i^h p_4$. The pseudolines $\ell(x_{i-1}, p_4)$ and $\ell(x_{i+2}, p_1)$ divide the plane into four quadrants, each containing one of the pseudosegments conv$\{p_1, p_4\}$, conv$\{p_4, x_{i+2}\}$, conv$\{x_{i+2}, x_{i-1}\}$, conv$\{x_{i-1}, p_1\}$, in this cyclic order. By the ordering $\prec_i^h$, p_2 and p_3 are contained in the quadrant containing conv$\{p_1, p_4\}$. Furthermore, the pseudoline $\ell(p_2, p_3)$ must cross this quadrant, entering the boundary ray containing p_1, then meeting p_2 before p_3 and finally exiting the boundary ray containing p_4. If both (p_1, p_2, p_3) and (p_2, p_3, p_4) are outer (inner) convex chains, it follows by the observations above that conv$\{p_1, p_4\}$ and conv$\{x_{i-1}, x_{i+2}\}$ are separated by (lie on the same side of) $\ell(p_2, p_3)$. This implies that conv$\{x_{i-1}, x_{i+2}\}$ and p_4 are separated by (lie on the same side of) $\ell(p_1, p_2)$ and $\ell(p_1, p_3)$. Hence, (p_1, p_2, p_4) and (p_1, p_3, p_4) are both outer (inner) convex chains, which proves part (i). By Carathéodory's theorem, it suffices to check part (ii) for inner and outer convex chains $p_1 \prec_i^h p_2 \prec_i^h p_3 \prec_i^h p_4$. However, it follows from the discussion above that $\ell(p_1, p_4)$ does not intersect conv$\{p_2, p_3\}$ and that $\ell(p_2, p_3)$ does not intersect conv$\{p_1, p_4\}$. As in the proof of Observation 3.5, we obtain that $\{p_1, p_2, p_3, p_4\}$ is in convex position. See Fig. 5. □

Letting c_i and d_i denote the length of the longest inner convex chain and the length of the longest outer convex chain in P_i, respectively, applying Theorem 2.1
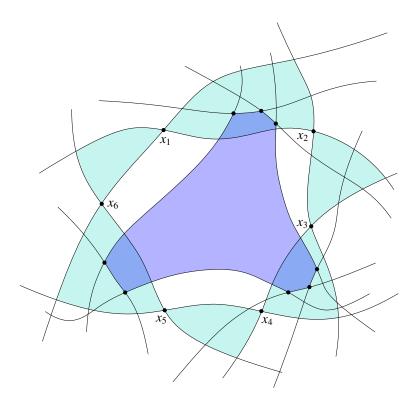
FIGURE 6. Joining inner convex chains $A_1 \subseteq P_1, A_3 \subseteq P_3$, and $A_5 \subseteq P_5$ to form a subset in convex position (convex hull in darker shade).

to the longest horizontal chain in P_i and using Observation 3.6, we obtain

$$(5) \qquad\qquad h_i \leq \binom{c_i + d_i - 2}{c_i - 1}.$$

Observation 3.7. *Suppose that $k \geq 4$ is even, and let $A_1 \subseteq P_1$, $A_2 \subseteq P_2$, ..., $A_k \subseteq P_k$. If each A_i is an inner convex chain, then $A_1 \cup A_3 \cup \cdots \cup A_{k-1}$ is in convex position, and so is $A_2 \cup A_4 \cup \cdots \cup A_k$.*

Proof. The proof follows in the same way as Observation 3.5, and we repeatedly use the fact mentioned at the beginning of that proof. It suffices to prove that any four points $p_1, p_2, p_3, p_4 \in A_1 \cup A_3 \cup \cdots \cup A_{k-1}$ are in convex position. If all the points lie in one chain, we are done. Consider the case where three points belong to the same chain, say $p_1, p_2, p_3 \in A_{i_1}$ and $p_4 \in A_{i_2}$ with $i_1 \neq i_2$. By Observation 3.4, x_{i_1-1} and p_4 belong to the same open region determined by the pseudolines $\ell(p_1, p_2), \ell(p_1, p_3), \ell(p_2, p_3)$. Therefore, by the last statement of Observation 2.3(ii), the convexity of $\{p_1, p_2, p_3, x_{i_1-1}\}$ implies that $\{p_1, p_2, p_3, p_4\}$ is in convex position.

If one of the chains contains exactly two of our points, say $p_1, p_2 \in A_i$, then neither p_1 nor p_2 can be in the convex hull of the other three points, as Observation 3.4 implies that the pseudoline $\ell(p_1, p_2)$ does not intersect the pseudosegment conv$\{p_3, p_4\}$.

To finish the proof, we need to verify that if one of the chains contains exactly one of our points, say $p_1 \in A_i$, then p_1 is not in the convex hull of the other three points. This follows from the fact that $\ell(x_i, x_{i+1})$ separates p_1 from p_2, p_3 and p_4. $\qquad\square$

Proof of Theorem 1.3. . Let P be a pseudo-configuration, and suppose that P does not contain n points in convex position. Let k be an even integer to be specified later, and let $X = \{x_1, x_2, \ldots, x_k\} \subseteq P$ be a subset in convex position whose existence is guaranteed by Theorem 2.4. Define $P_i, v_i, h_i, a_i, b_i, c_i, d_i$, as above.

By Observation 3.6(ii), we have $d_i < n$. By Observation 3.5, we have

$$(6) \qquad\qquad b_i + a_{i+1} < n$$

for all i, and, by Observation 3.7,

$$(7) \qquad\qquad c_1 + c_2 + \cdots + c_k < 2n.$$

Combining these with inequalities (2)–(5), we obtain

$$
\begin{aligned}
\frac{N^k}{2^{8k^2}} &\leq \prod_{i=1}^{k} |P_i| \\
&\leq \prod_{i=1}^{k} v_i h_i \\
&\leq \prod_{i=1}^{k} \binom{a_i + b_i - 2}{a_i - 1}\binom{c_i + d_i - 2}{d_i - 1} \\
&\leq \prod_{i=1}^{k} 2^{a_i+b_i} d_i^{c_i} < 2^{kn+2n\log n},
\end{aligned}
$$

which gives us

$$N < 2^{n+\frac{2n\log n}{k}+8k}.$$

Setting k to be the smallest even integer greater than or equal to $\sqrt{n \log n}/2$, gives the estimate

$$N = O\left(2^{n+8\sqrt{n\log n}}\right). \qquad\square$$

Remark. With a less wasteful computation, in particular by using the estimate above in place of Theorem 1.2 in the proof of Theorem 2.4, the constant 8 in the exponent can be replaced by $4\sqrt{2} + \epsilon$.

REFERENCES

[1] I. Bárány and P. Valtr, A positive fraction Erdős–Szekeres theorem, *Discrete Comput. Geom.*, 19:335–342, 1998.

[2] T. Bisztriczky and G. Fejes Tóth, A generalization of the Erdős–Szekeres convex n-gon theorem, *J. Reine Angew. Math.*, 395:167–170, 1989.

[3] T. Bisztriczky and G. Fejes Tóth, Nine convex sets determine a pentagon with convex sets as vertices, *Geom. Dedicata*, 31:89–104, 1989.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented matroids*, Encyclopedia of Mathematics and its Applications. Cambridge University Press, second edition, 1999.

[5] R. Dhandapani, J. E. Goodman, A. Holmsen, R. Pollack, and S. Smorodinsky, Convexity in topological affine planes, *Discrete Comput. Geom.*, 38: 243–257, 2007.

[6] M. G. Dobbins, A. F. Holmsen, and A. Hubard, The Erdős–Szekeres problem for non-crossing convex sets, *Mathematika*, 60: 463–484, 2014.

[7] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.*, 2:463–470, 1935.

[8] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 3–4:53–62, 1960/1961.

[9] J. Fox, J. Pach, B. Sudakov, and A. Suk, Erdős–Szekeres-type theorems for monotone paths and convex bodies, *Proc. London Math. Soc.*, 105:953–982, 2012.

[10] J. E. Goodman, Pseudoline arrangements, In *Handbook of discrete and computational geometry, 2nd Edition*, CRC Press Ser. Discrete Math. Appl., pages 97–128. CRC, Boca Raton, FL, 2004.

[11] J. E. Goodman and R. Pollack, A combinatorial perspective on some problems in geometry, In *Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. I (Baton Rouge, La., 1981)*, volume 32, pages 383–394, 1981.

[12] A. Hubard, L. Montejano, E. Mora, and A. Suk, Order types of convex bodies, *Order*, 28:121–130, 2011.

[13] D. E. Knuth, *Axioms and hulls*, Lecture Notes in Computer Science. Springer-Verlag, Berlin, 1992.

[14] G. Moshkovitz and A. Shapira, Ramsey Theory, integer partitions and a new proof of the Erdős–Szekeres Theorem, *Adv. Math.* 262: 1107–1129, 2014.

[15] J. Pach and J. Solymosi, Canonical theorems for convex sets, *Discrete Comput. Geom.*, 19:427–435, 1998.

[16] J. Pach and G. Tóth, A generalization of the Erdős–Szekeres theorem to disjoint convex sets, *Discrete Comput. Geom.*, 19:437–445, 1998.

[17] J. Pach and G. Tóth, Erdős–Szekeres-type theorems for segments and noncrossing convex sets, *Geom. Dedicata*, 81:1–12, 2000.

[18] A. Pór and P. Valtr, The partitioned version of the Erdős–Szekeres theorem, *Discrete Comput. Geom.*, 28:625–637, 2002.

[19] A. Suk, On the Erdős–Szekeres convex polygon problem, *J. Am. Math. Soc.* 30: 1047–1053, 2017.

ANDREAS F. HOLMSEN, DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, DAEJEON, SOUTH KOREA.
E-mail address: andreash@kaist.edu

HOSSEIN NASSAJIAN MOJARRAD, DEPARTMENT OF MATHEMATICS, EPFL, LAUSANNE, SWITZERLAND.
E-mail address: hossein.mojarrad@epfl.ch

JÁNOS PACH, EPFL, LAUSANNE, SWITZERLAND AND RÉNYI INSTITUTE, BUDAPEST, HUNGARY.
E-mail address: pach@cims.nyu.edu

GÁBOR TARDOS, RÉNYI INSTITUTE, HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, HUNGARY.
E-mail address: tardos@renyi.hu