A. PAC learning

1. Give a PAC-learning algorithm for the concept class $C$ formed by closed intervals $[a, b]$ with $a, b \in \mathbb{R}$. You should give a careful proof as for the case of axis-aligned rectangles described in class.

2. Give a PAC-learning algorithm for the concept class $C_2$ formed by unions of two closed intervals, that is $[a, b] \cup [c, d]$, with $a, b, c, d \in \mathbb{R}$. Extend your result to derive a PAC-learning algorithm for the concept class $C_p$ formed by unions of $p \geq 1$ closed intervals, thus $[a_1, b_1] \cup \cdots \cup [a_p, b_p]$, with $a_k, b_k \in \mathbb{R}$ for $k \in [1, p]$. What are the time and sample complexities of your algorithm as a function of $p$?

B. Rademacher complexity, growth function

1. Let $H$ be the family of threshold functions over the real line: $H = \{ x \mapsto 1_{x \leq \theta} : \theta \in \mathbb{R} \} \cup \{ x \mapsto 1_{x > \theta} : \theta \in \mathbb{R} \}$. Give an upper bound on the growth function $\Pi_H(m)$. Use that to derive an upper bound on $\mathfrak{R}_m(H)$.

2. Let $H_1$ and $H_2$ be two families of functions mapping $\mathcal{X}$ to $\{0, 1\}$ and let $H = \{ h_1 h_2 : h_1 \in H_1, h_2 \in H_2 \}$. Show that the empirical Rademacher complexity of $H$ for any sample $S$ of size $m$ can be bounded as follows:

$$\hat{\mathfrak{R}}_S(H) \leq \hat{\mathfrak{R}}_S(H_1) + \hat{\mathfrak{R}}_S(H_2).$$

Hint: use the Lipschitz function $x \mapsto \max(0, x - 1)$ and Talagrand’s contraction lemma (see textbook or slides).

Use that to bound the Rademacher complexity $\mathfrak{R}_m(U)$ of the family $U$ of intersections of two concepts $c_1$ and $c_2$ with $c_1 \in C_1$ and $c_2 \in C_2$ in terms of the Rademacher complexities of $C_1$ and $C_2$. 