Speech Recognition
Lecture 3: Weighted Transducer Algorithms

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This Lecture

- Shortest-distance algorithms
- Composition
- Epsilon-removal
- Determinization
- Pushing
- Minimization

Shortest-Distance Problem

- Definition: for any regulated weighted transducer T, define the shortest distance from state q to F as

$$d(q, F) = \bigoplus_{\pi \in P(q,F)} w[\pi].$$

- Problem: compute $d(q, F)$ for all states $q \in Q$.
- Algorithms:
  - Generalization of Floyd-Warshall.
  - Single-source shortest-distance algorithm.

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  - Single-source shortest-distance algorithm.

All-Pairs Shortest-Distance Algorithm

(MM, 2002)

- Assumption: closed semiring (not necessarily idempotent).
- Idea: generalization of Floyd-Warshall algorithm.
- Properties:
  - Time complexity: $\Omega(|Q|^3(T_\oplus + T_\otimes + T_\star))$.
  - Space complexity: $\Omega(|Q|^2)$ with an in-place implementation.

Closed Semirings

(Lehmann, 1977)

- Definition: a semiring is closed if the closure is well defined for all elements and if associativity, commutativity, and distributivity apply to countable sums.
- Examples:
  - Tropical semiring.
  - Probability semiring when including infinity or when restricted to well-defined closures.

Pseudocode

$\text{Gen-All-Pairs}(G)$

$\textbf{for } i \leftarrow 1 \textbf{ to } |Q| \textbf{ do}$
$\quad \textbf{for } j \leftarrow 1 \textbf{ to } |Q| \textbf{ do}$
$\quad\quad d[i,j] \leftarrow \bigoplus_{e \in E \cap P(i,j)} w[e]$
$\textbf{for } k \leftarrow 1 \textbf{ to } |Q| \textbf{ do}$
$\quad \textbf{for } i \leftarrow 1 \textbf{ to } |Q|, i \neq k \textbf{ do}$
$\quad\quad \textbf{for } j \leftarrow 1 \textbf{ to } |Q|, j \neq k \textbf{ do}$
$\quad\quad\quad d[i,j] \leftarrow d[i,j] \oplus (d[i,k] \otimes d[k,k]^* \otimes d[k,j])$
$\quad \textbf{for } i \leftarrow 1 \textbf{ to } |Q|, i \neq k \textbf{ do}$
$\quad\quad d[k,i] \leftarrow d[k,k]^* \otimes d[k,i]$
$\quad\quad d[i,k] \leftarrow d[i,k] \otimes d[k,k]^*$
$\quad d[k,k] \leftarrow d[k,k]^*$

Single-Source Shortest-Distance Algorithm

(MM, 2002)

- Assumption: k-closed semiring.

$$\forall x \in \mathbb{K}, \ \bigoplus_{i=0}^{k+1} x^i = \bigoplus_{i=0}^{k} x^i.$$

- Idea: generalization of relaxation, but must keep track of weight added to $d[q]$ since the last time q was enqueued.

- Properties:
  - works with any queue discipline and any k-closed semiring.
  - Classical algorithms are special instances.

Pseudocode

```
GENERIC-SINGLE-SOURCE-SHORTEST-DISTANCE (G, s)
for i ← 1 to |Q|
    do d[i] ← r[i] ← 0̄
d[s] ← r[s] ← 1̄
S ← {s}
while S ≠ ∅
    do q ← head(S)
       DEQUEUE(S)
       r′ ← r[q]
       r[q] ← 0̄
       for each e ∈ E[q]
           do if d[n[e]] ≠ d[n[e]] ⊕ (r′ ⊗ w[e])
                then d[n[e]] ← d[n[e]] ⊕ (r′ ⊗ w[e])
                     r[n[e]] ← r[n[e]] ⊕ (r′ ⊗ w[e])
                     if n[e] ∉ S
                        then ENQUEUE(S, n[e])
d[s] ← 1̄
```

Notes

- Complexity:
  - depends on queue discipline used.

$$O(|Q| + (T_\oplus + T_\otimes + C(A))|E| \max_{q \in Q} N(q) + (C(I) + C(E)) \sum_{q \in Q} N(q))$$

  - coincides with that of Dijkstra and Bellman-Ford for shortest-first and FIFO orders.
  - linear for acyclic graphs using topological order.

$$O(|Q| + (T_\oplus + T_\otimes)|E|)$$

- Approximation: ϵ-k-closed semiring, e.g., for graphs in probability semiring.

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Composition

- **Definition**: given two weighted transducer T_1 and T_2 over a commutative semiring, the composed transducer $T = T_1 \circ T_2$ is defined by

$$(T_1 \circ T_2)(x, y) = \bigoplus_{z} T_1(x, z) \otimes T_2(z, y).$$

- **Algorithm**:
  - Epsilon-free case: matching transitions.
  - General case: ϵ-filter transducer.
  - Complexity: quadratic, $O(|T_1||T_2|)$.
  - On-demand construction.

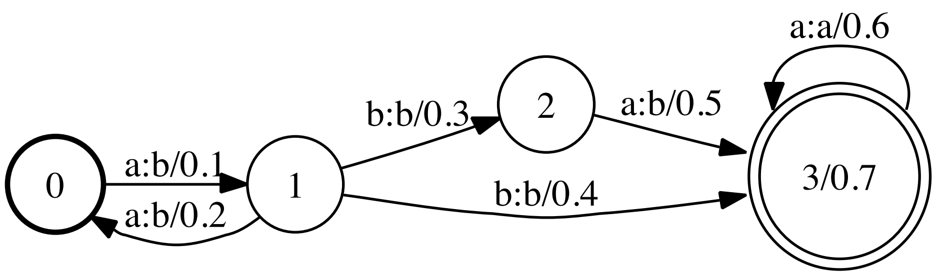
Epsilon-Free Composition

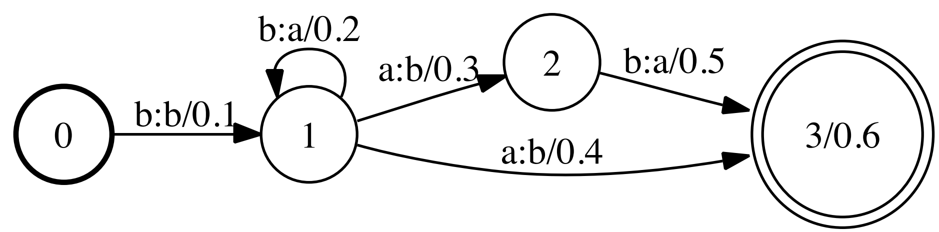
- States $Q \subseteq Q_1 \times Q_2$.
- Initial states $I = I_1 \times I_2$.
- Final states $F = Q \cap F_1 \times F_2$.
- Transitions

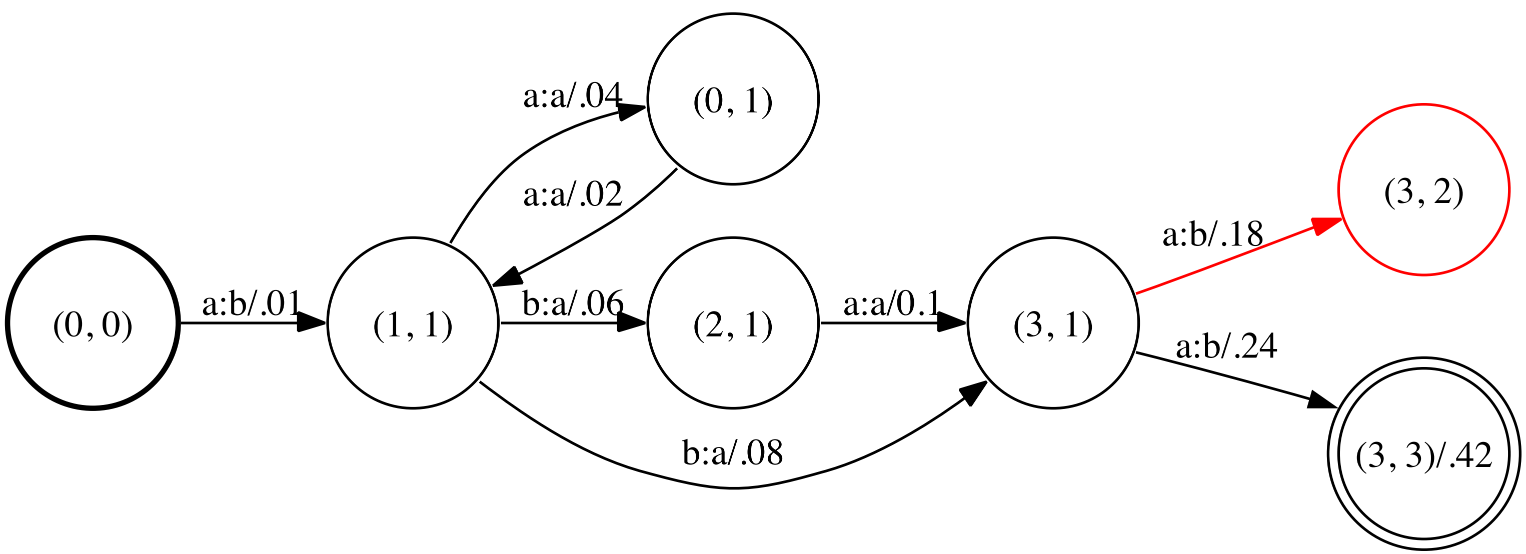
$$E = \{((q_1, q_1'), a, c, w_1 \otimes w_2, (q_2, q_2')) : \\ (q_1, a, b, w_1, q_2), (q_1', b, c, w_2, q_2') \in Q\}.$$

Illustration

- **Program:** fsmcompose A.fsm B.fsm >C.fsm
  fstcompose A.fsm B.fsm >C.fsm

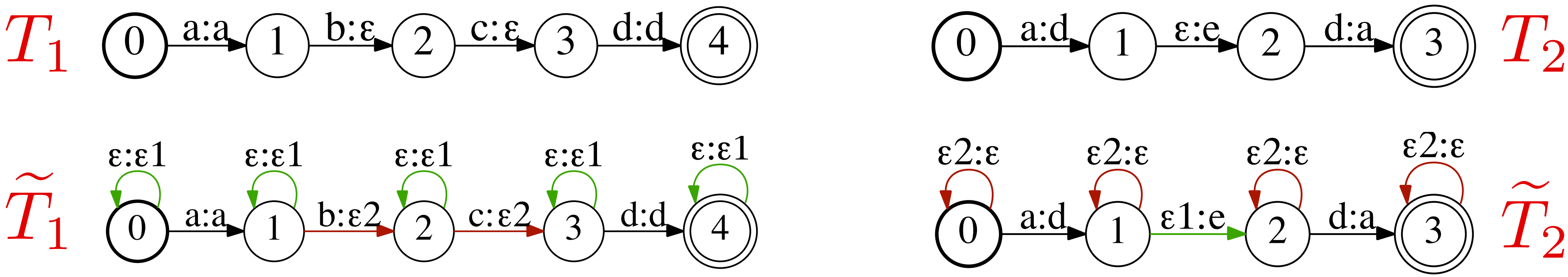


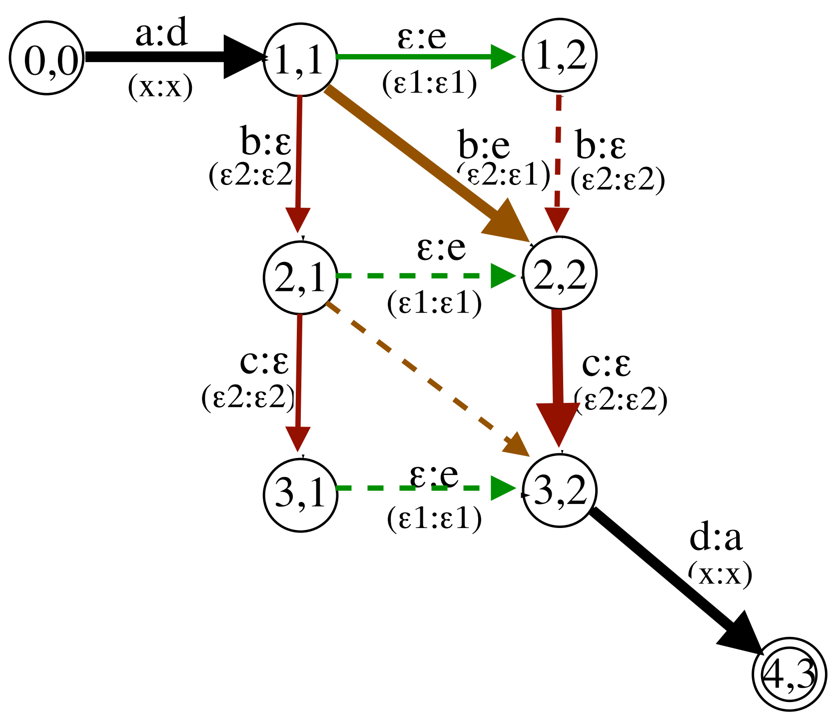


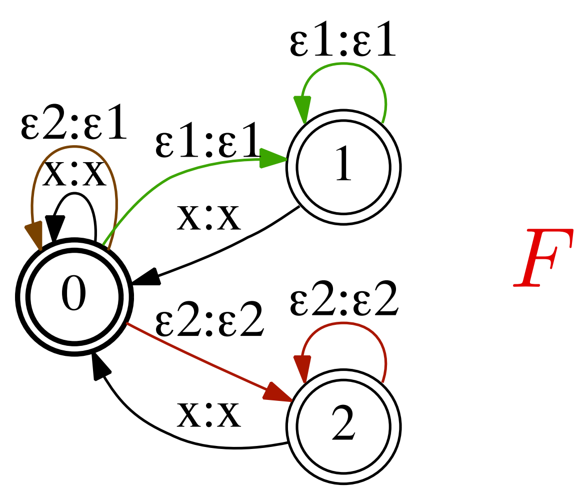


Redundant ε-Paths Problem

(MM et al. 1996)



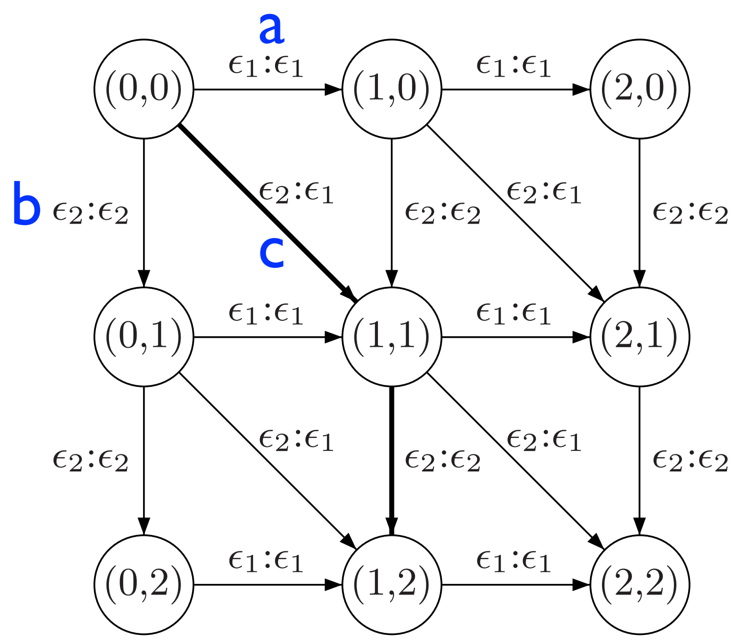




$$T = \widetilde{T}_1 \circ F \circ \widetilde{T}_2.$$

Correctness of Filter

- **Proposition**: filter F allows a unique path between two states of the following grid.



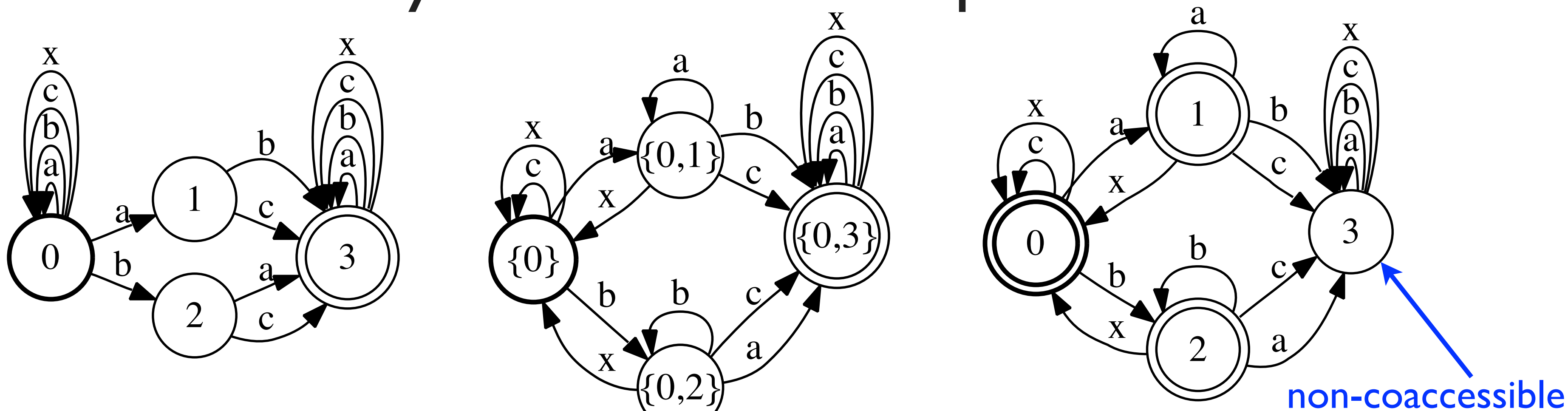
- **Proof**: Observe that a necessary and sufficient condition is that the following sequences be forbidden: ab, ba, ac, and bc.

Correctness of Filter

- Proof (cont.): Let $\sigma = \{a, b, c, x\}$, then set of sequences forbidden is exactly

$$L = \sigma^*(ab + ba + ac + bc)\sigma^*.$$

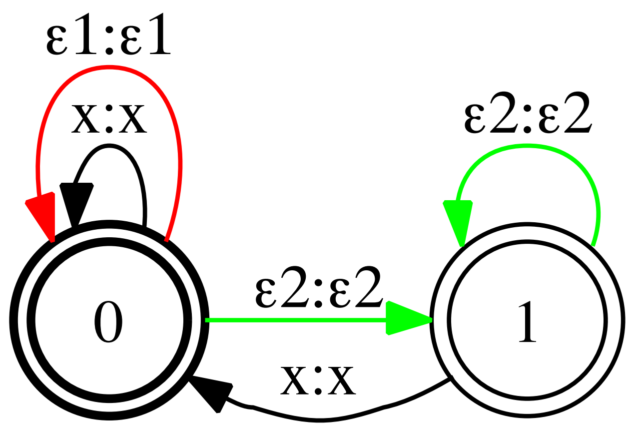
- An automaton representing the complement can be constructed by determ. and complementation.



A $\quad$ $det(A)$ $\quad$ $\overline{det(A)}$

Other Filters

(Pereira and Riley, 1997)



Sequential Filter.

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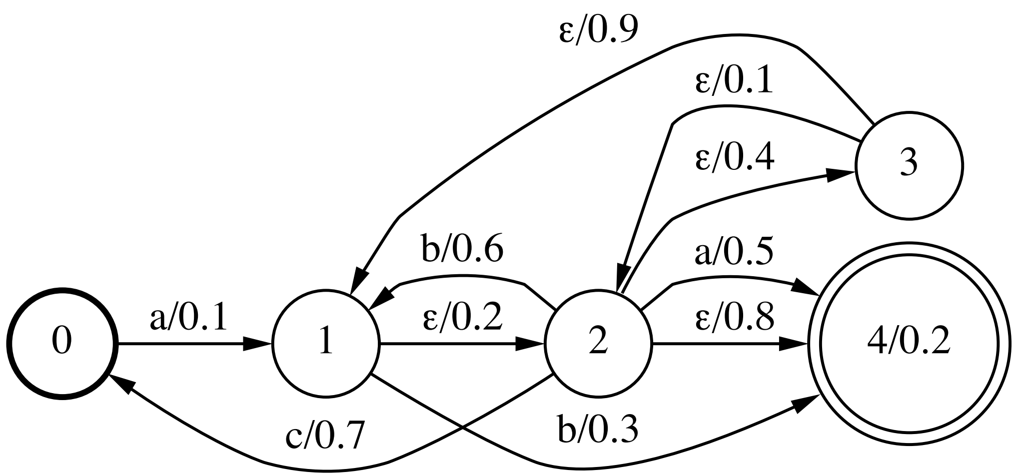
Epsilon-Removal

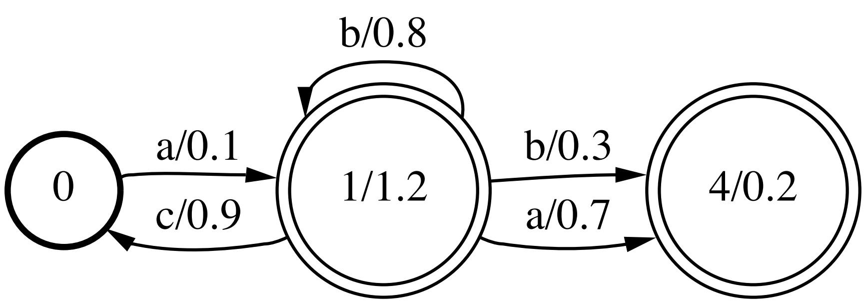
- **Definition**: given weighted transducer T, create equivalent weighted transducer with no epsilon-transition.
- **Algorithm components**:
  - Computation of the ε-closure at each state:

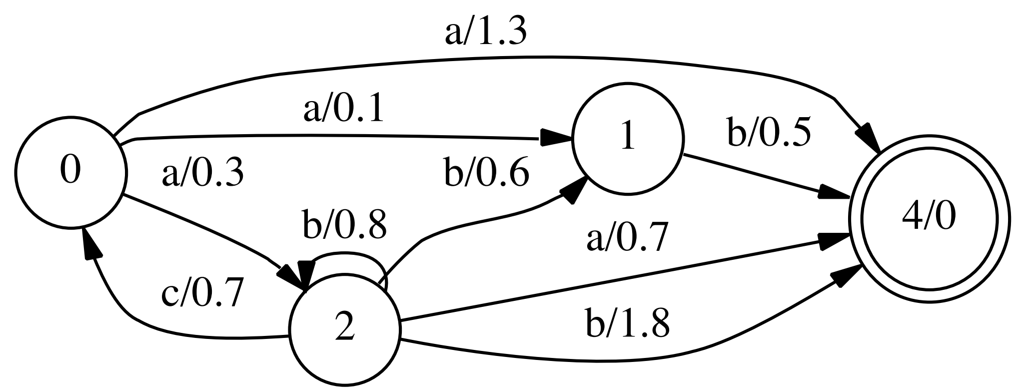
$$C[p] = \left\{ (q, d_\epsilon[p, q]) : d_\epsilon[p, q] \neq \overline{0}) \right\} \text{ with } d_\epsilon[p, q] = \bigoplus_{\pi \in P(p, \epsilon, q)} w[\pi].$$

  - Removal of εs.
  - On-demand construction.

Illustration







Main Algorithm

- Shortest-distance algorithms:
  - closed semirings: generalization of Floyd-Warshall algorithm.
  - *k*-closed semirings: single-source shortest-distance algorithm.
- Complexity: shortest-distance and removal.
  - Acyclic T_ϵ: $O(|Q|^2 + |Q||E|(T_\oplus + T_\otimes))$.
  - General case, tropical semiring:

$$O(|Q||E| + |Q|^2 \log |Q|).$$

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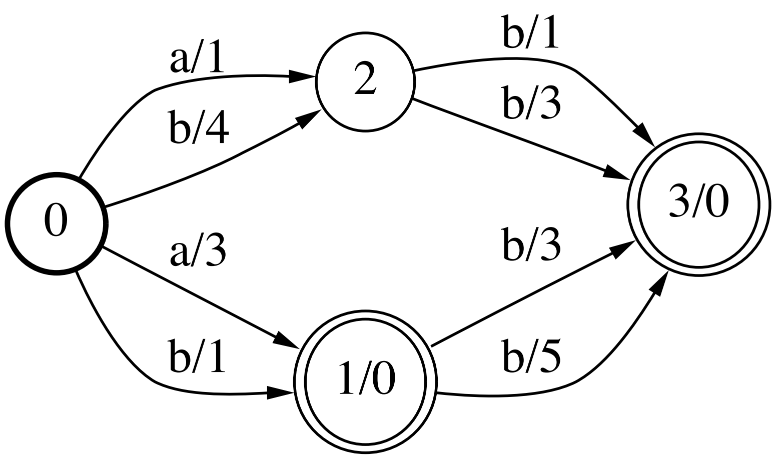
Determinization

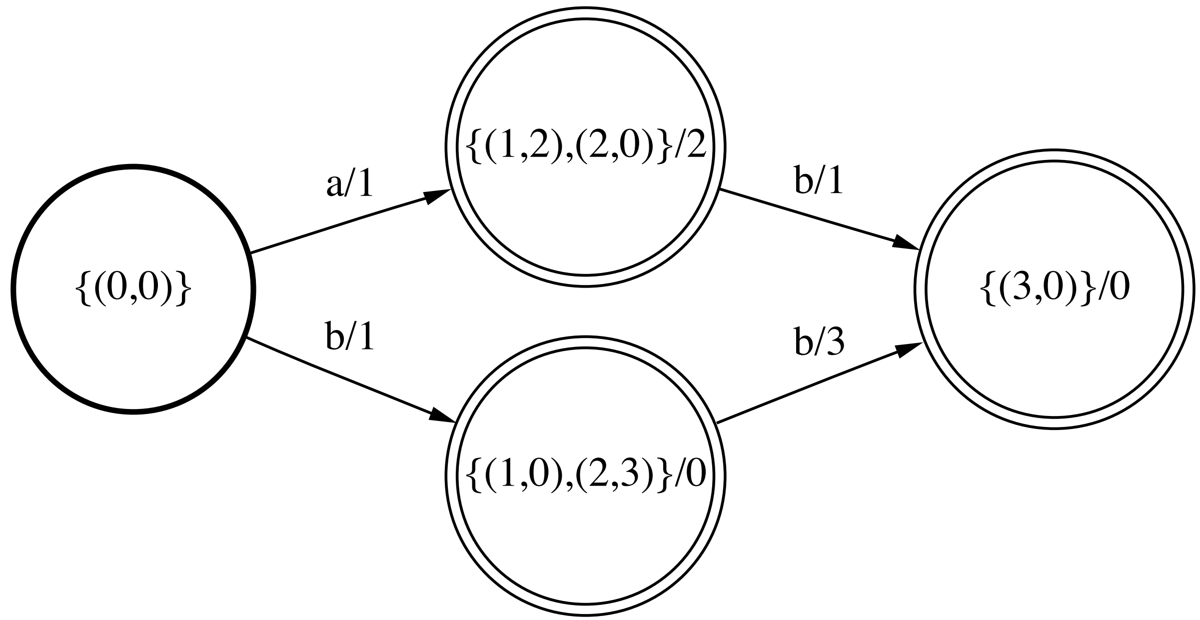
- **Definition**: given weighted transducer T, create equivalent deterministic weighted transducer.
- **Algorithm** (weakly left divisible semirings):
  - generalization of subset constructions to weighted labeled subsets

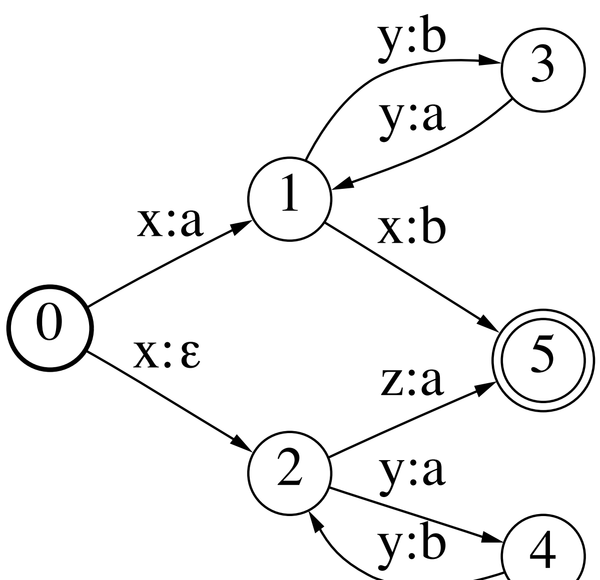
$$\{(q_1, x_1, w_1), \ldots, (q_m, x_m, w_m)\}.$$

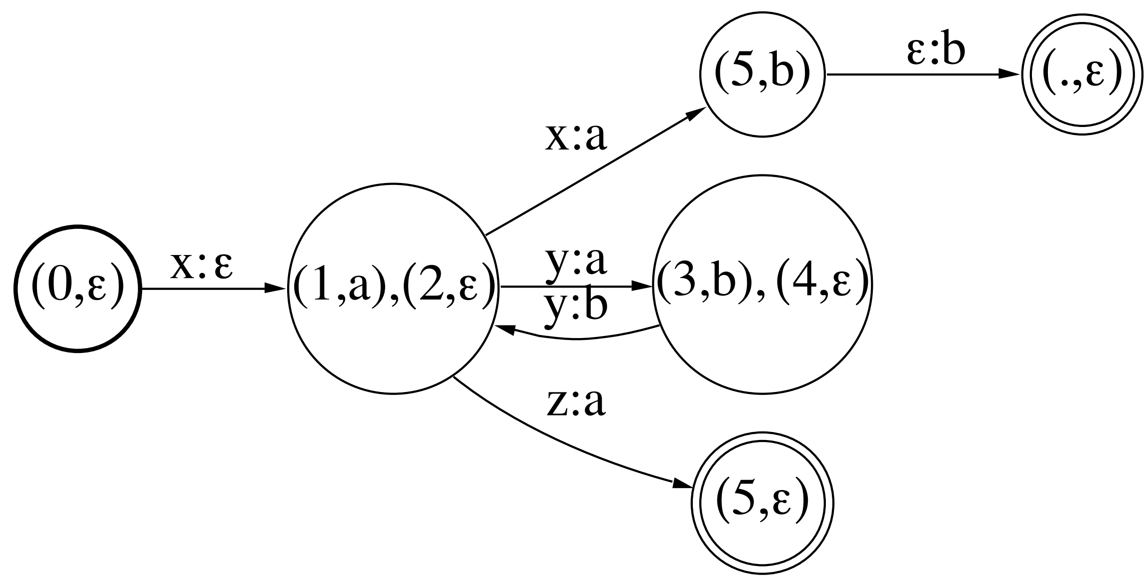
  - complexity: exponential, but lazy implementation.
  - not all weighted transducers are determinizable but all acyclic weighted transducers are. Test? For some cases, using the twins property.

Illustration

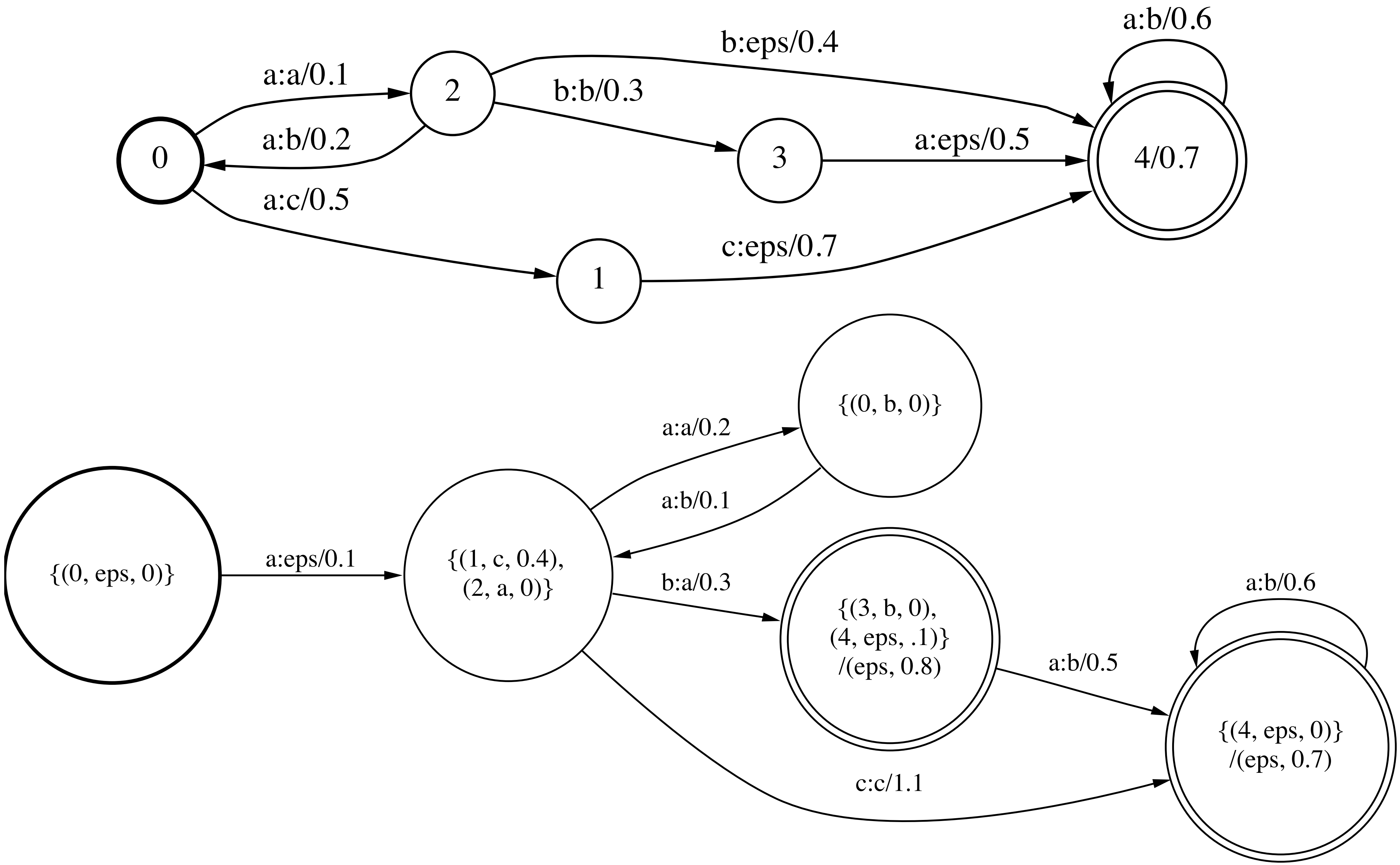




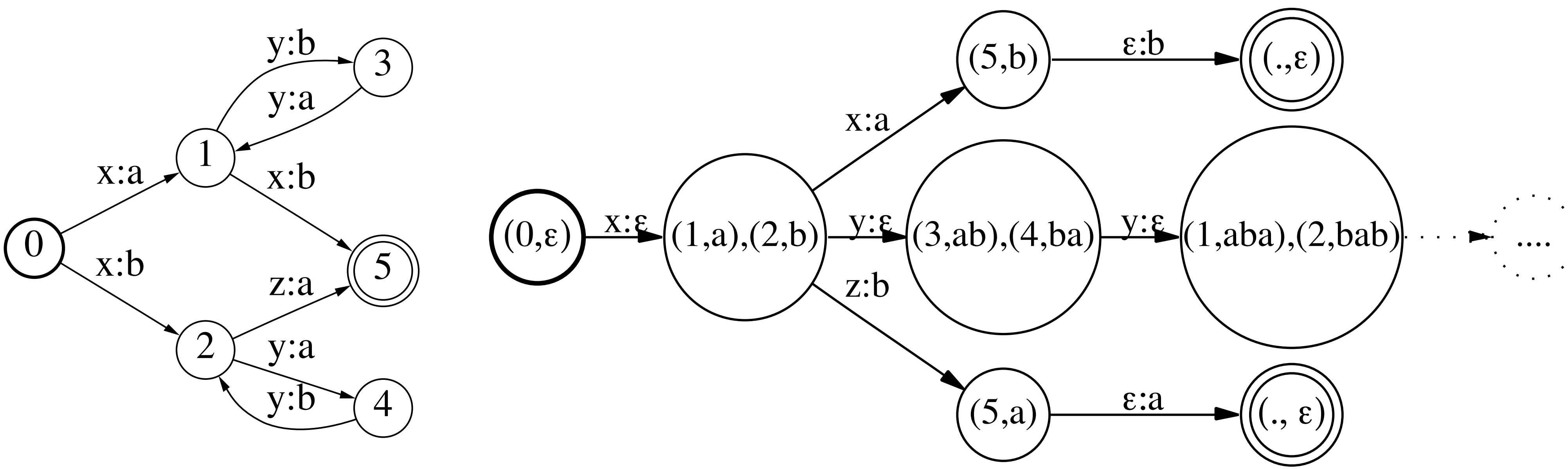




Illustration



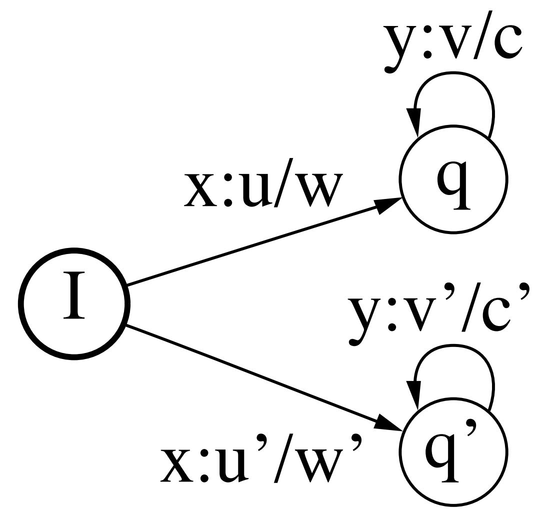
Non-Determinizable Transducer

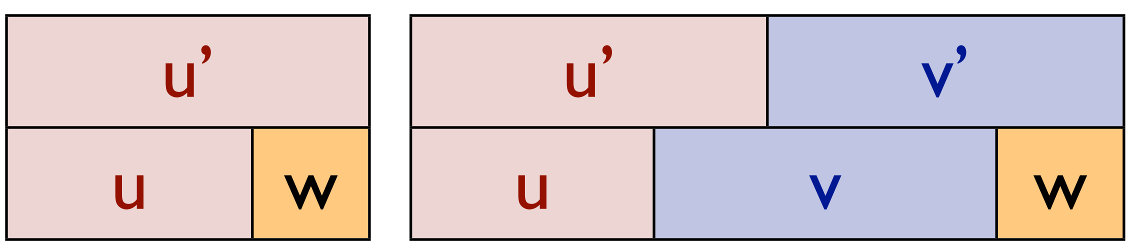


Twins Property

(Choffrut, 1978; MM 1997)

- Definition: a weighted transducer T over the tropical semiring has the twins property if for any two states q and q' as in the figure, the following holds:
  - $c = c'$;
  - $u^{-1}u' = (uv)^{-1}u'v'$.





Determinizability

(Choffrut, 1978; MM 1997; Allauzen and MM, 2002)

- Theorem: a trim unambiguous weighted automaton over the tropical semiring is determinizable iff it has the twins property.
- Theorem: let T be a weighted transducer over the tropical semiring. Then, if T has the twins property, then it is determinizable.
- Algorithm for testing the twins property:
  - unambiguous automata: $O(|Q|^2 + |E|^2)$.
  - unweighted transducers: $O(|Q|^2(|Q|^2 + |E|^2))$.

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Pushing

(MM, 1997; 2004)

- **Definition**: given weighted transducer T, create equivalent weighted transducer such the sum (longest common prefix) of the weights (output strings) of all outgoing paths be $\overline{1}$ (ε) at all states, modulo initial states.

- **Algorithm components**:
  - Single-source shortest-distance computation

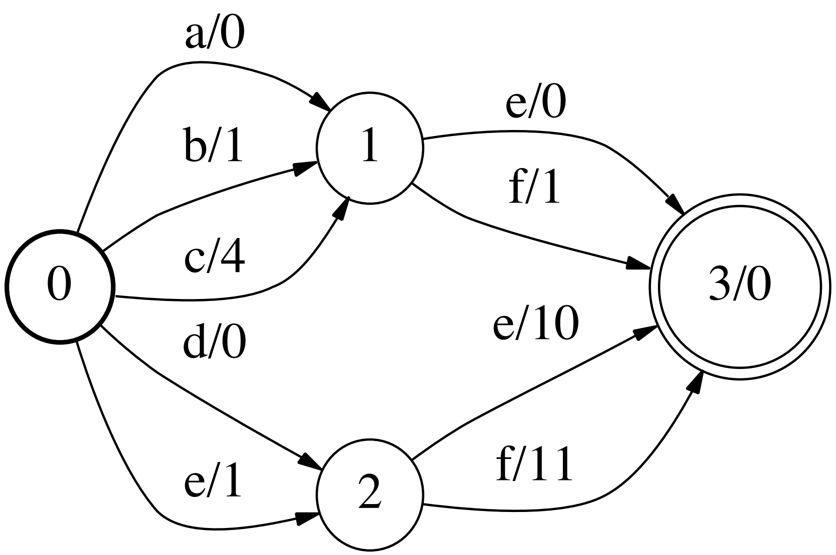
$$d[q] = \bigoplus_{\pi \in P(q,F)} w[\pi].$$

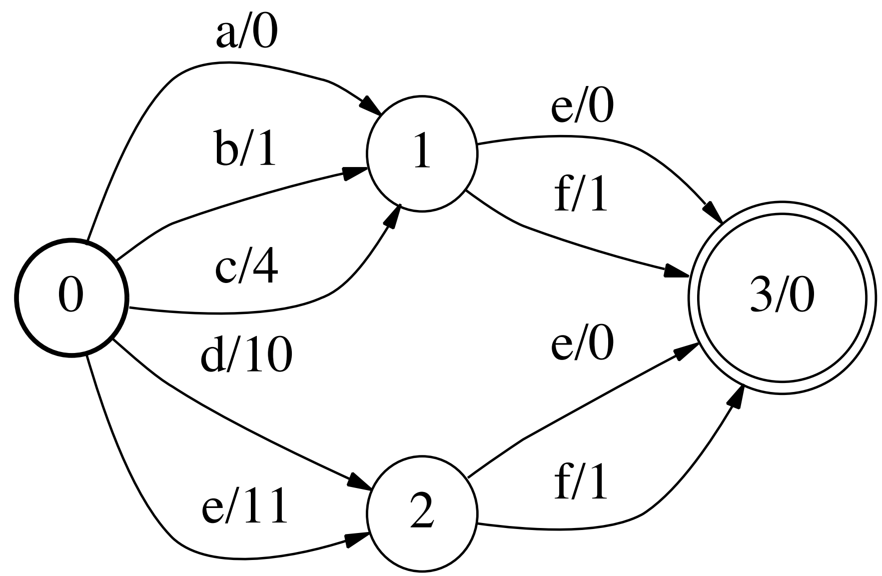
  - Reweighting: $w[e] \leftarrow (d[p[e]])^{-1}(w[e] \otimes d[n[e]])$ for each transition e.

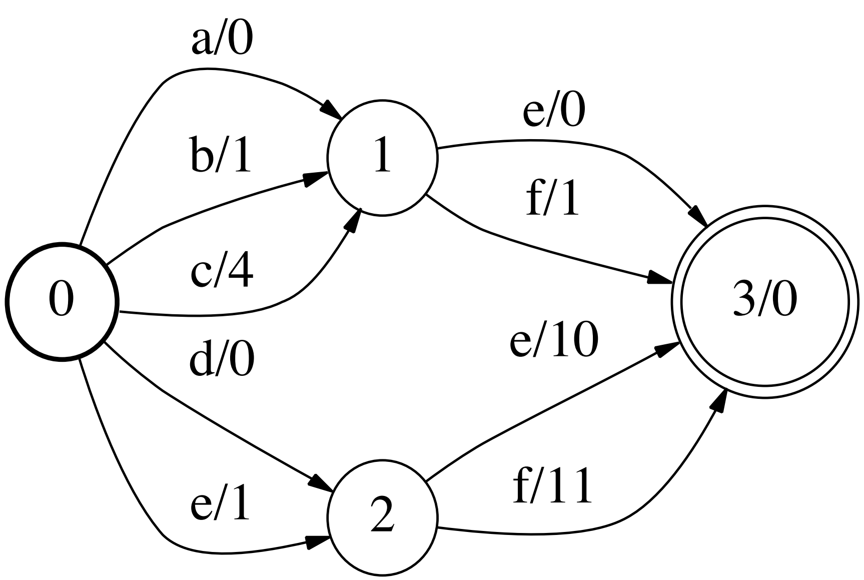
Main Algorithm

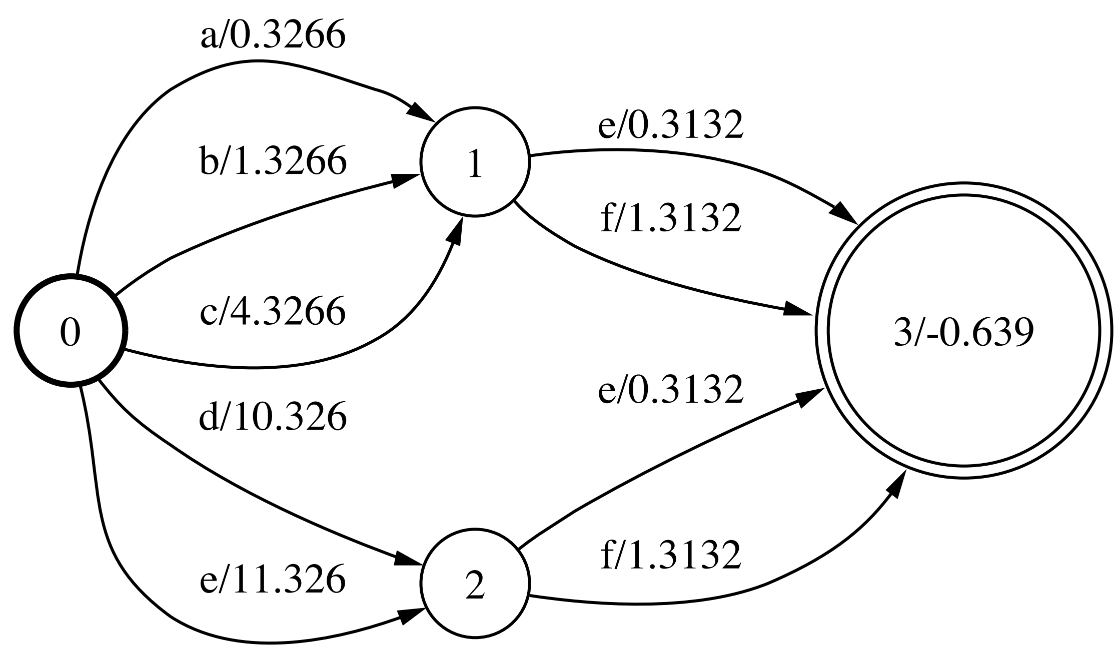
- Automata: single-source shortest-distance.
  - acyclic case: $O(|Q| + |E|(T_\oplus + T_\otimes))$.
  - general case tropical semiring: $O(|Q| \log |Q| + |E|)$.
  - general case *k*-closed semirings
    $$O(|Q| + (T_\oplus + T_\otimes + C(A))|E| \max_{q \in Q} N(q) + (C(I) + C(E)) \sum_{q \in Q} N(q))$$
  - general case closed semirings $\Omega(|Q|^3(T_\oplus + T_\otimes + T_\star))$.
- Transducers: $O((|P_{max}| + 1)\,|E|)$.

Illustration

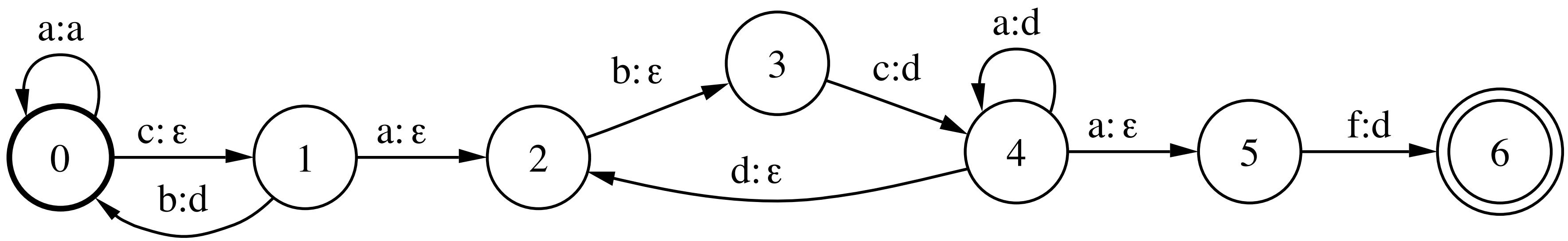


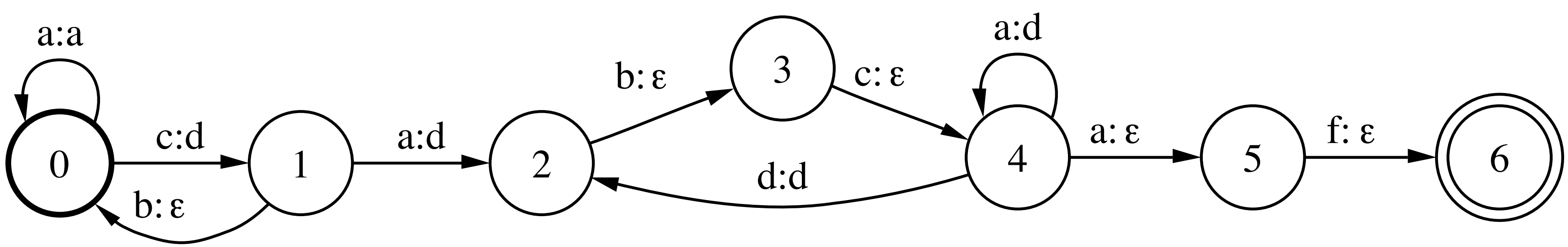






Illustration





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Minimization

(MM, 1997, 2000, 2005)

- **Definition**: given deterministic weighted transducer T, create equivalent deterministic weighted transducer with the minimal number of states (and transitions).

- **Algorithm components**:
  - apply pushing to create canonical representation.
  - apply unweighted automata minimization after encoding (input labels, output label, weight) as a single label.

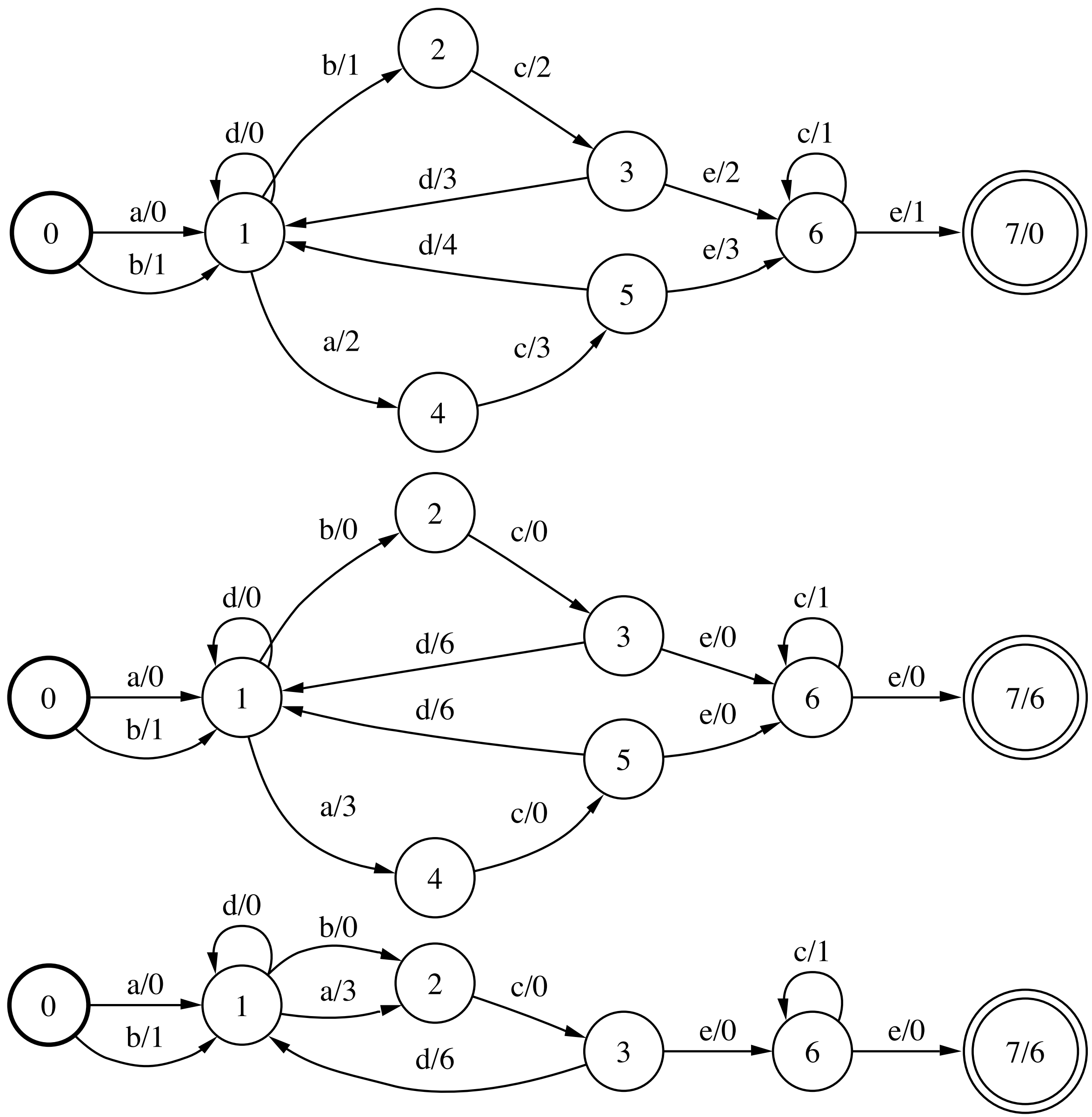
Algorithm

(MM, 1997, 2000, 2005)

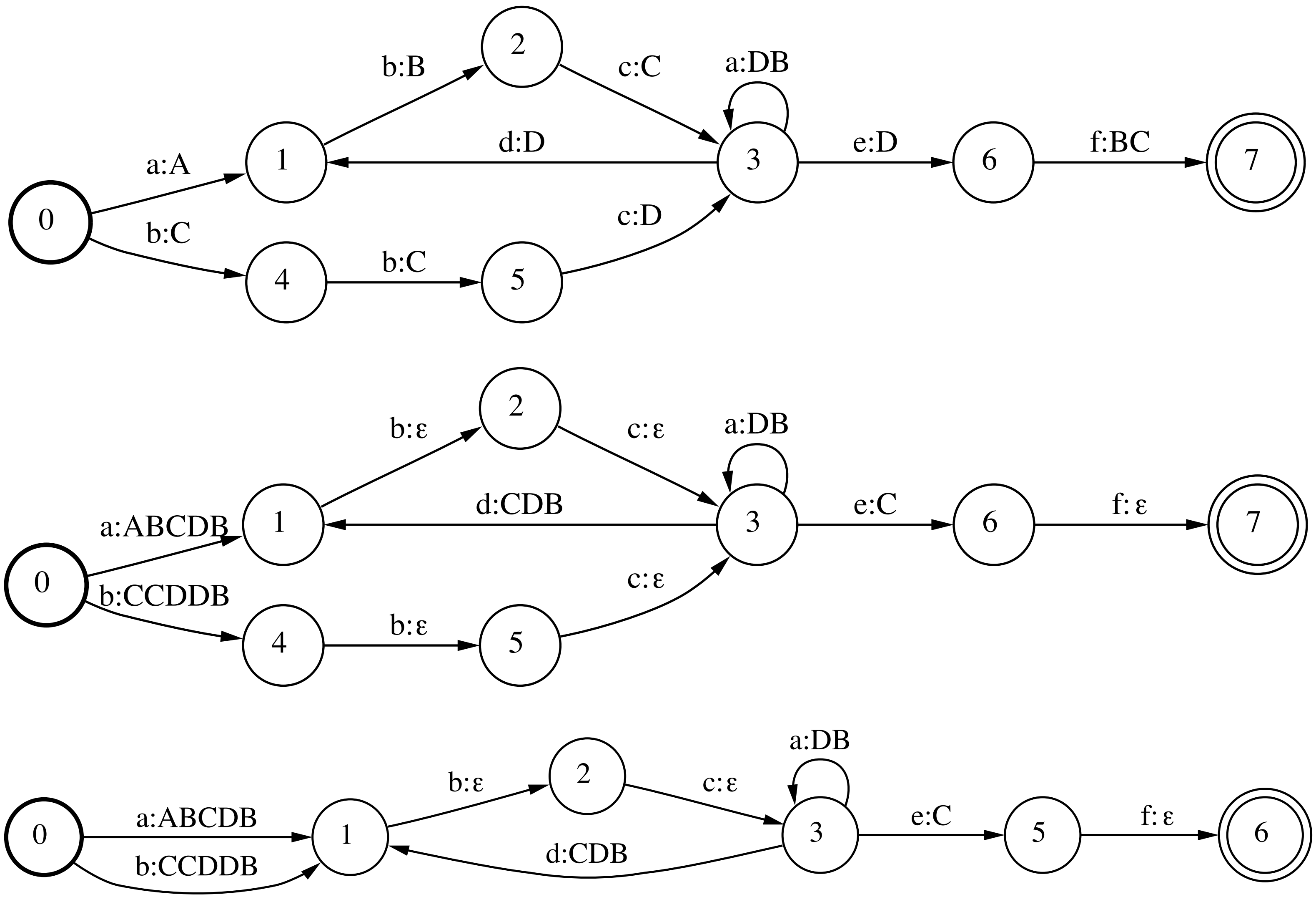
- Automata: pushing and automata minimization, general (Hopcroft, 1971) and acyclic case (Revuz 1992).
  - acyclic case: $O(|Q| + |E|(T_{\oplus} + T_{\otimes}))$.
  - general case tropical semiring: $O(|E| \log |Q|)$.
- Transducers:
  - acyclic case: $O(S + |Q| + |E| \, (|P_{max}| + 1))$.
  - general case tropical semiring:

$$O(S + |Q| + |E| \, (\log |Q| + |P_{max}|)).$$

Illustration



Illustration



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