A. Regular expressions

Let $\Sigma = \{a, b\}$. Give regular expressions describing the following languages:

1. The set of strings starting with $a$ and ending with $b$.
2. The set of strings containing exactly two $a$’s.
3. The set of strings containing exactly two consecutive $a$’s.
4. The set of strings that do not contain the sequence $ab$.

B. Regular languages

Let $\Sigma = \{a, b\}$. Show that the following languages are not regular:

1. $\{w \in \Sigma^* : w = w^R\}$.
2. $\{ww : w \in \Sigma^*\}$.

D. Determinism

Give an example of a minimal deterministic automaton such that the reverse is not deterministic.

C. Numbers in base 2

Describe the set of numbers in base 2 represented by the strings accepted by the automaton $A$ of Figure 1 (give a characterization and prove it).
E. Derivatives (bonus question)

Show that the transduction defined by:

\[ \tau: \Sigma^* \rightarrow \Sigma^* \]

\[ u \rightarrow u^{-1}L = \{v \in \Sigma^*: uv \in L\} \]

is rational if \( L \) is a regular language. To do so, you can proceed as follows.

1. Let \( A = (Q, I, F, E) \) be a deterministic automaton over the alphabet \( \Sigma \) accepting \( L \). For any \( u \in \Sigma^* \), let \( q \) be the state reached when reading \( u \) from the initial state of \( A \). Show that \( u^{-1}L \) is the set of strings labeling paths from \( q \) to \( F \).

2. Introduce a new alphabet \( \Sigma' \), that is \( \Sigma' \) contains exactly one symbol \( a' \) for each \( a \in \Sigma \). Let \( A' \) be the automaton obtained from \( A \) by adding a transition \((q, a', q)\) for each \((q, a, q') \in E\). Show that the language \( L' = L(A') \cap \Sigma^*\Sigma'^* \) is regular.

3. Let \( A'' \) be a deterministic automaton accepting \( L'' \). Use \( A'' \) to define a finite-state transducer realizing exactly the transduction \( \tau \).