A. Exponentially Weighted algorithm

The regret bound of the EW algorithm given in class does not match that of the Halving algorithm in the case where the loss of the best expert in hindsight is zero. In this problem, we will give a more favorable bound for such cases. We will adopt the same assumptions and use the same notation as for the regret bound theorem given in class for EW.

1. Fix $\eta > 0$. Show that for any $X \in [0, 1]$, $e^{-\eta X} \leq X(e^{-\eta} - 1) + 1$ \textit{(hint: use convexity of the exponential function)}. Use that to show that for a random variable $X$ taking values in $[0, 1]$,

$$\log E[e^{-\eta X}] \leq (e^{-\eta} - 1) E[X].$$ \hfill (1)

\textit{Solution:} Since $X$ is in $[0, 1]$, in view of $-\eta X = (1-X) \cdot 0 + X \cdot (-\eta)$, by convexity of exp, we can write

$$e^{-\eta X} \leq (1 - X) \cdot e^{0} + X \cdot e^{-\eta} = X(e^{-\eta} - 1) + 1.$$

Using this inequality and the inequality $\log(1 + x) \leq x$ valid for all $x > -1$, we can write

$$\log E[e^{-\eta X}] \leq \log \left(1 + E[X](e^{-\eta} - 1)\right) \leq E[X](e^{-\eta} - 1).$$

\hfill $\Box$

2. Prove that the cumulative loss of EW can be bounded as follows:

$$\sum_{t=1}^{T} L(\hat{y}_t, y_t) \leq \frac{\eta L_T^*}{1 - e^{-\eta}} + \log \frac{N}{1 - e^{-\eta}},$$ \hfill (2)

where $L_T^*$ is the loss of the best expert in hindsight \textit{(hint: use inequality (1) instead of Hoeffding’s inequality in the proof given in class)}. Compare this result with the Halving bound when $L_T^* = 0$ and $\eta$ large.
Solution: We use the potential $\Phi_t = \log \sum_{i=1}^{N} w_{t,i}$. Then, we can write

$$
\Phi_t - \Phi_{t-1} = \log \frac{\sum_{i=1}^{N} w_{t-1,i} e^{-\eta L(\hat{y}_{t,i}, y_t)}}{\sum_{i=1}^{N} w_{t-1,i}}
$$

$$
= \log \left( \mathbb{E}_{i \sim p_{t-1}} [e^{-\eta L(\hat{y}_{t,i}, y_t)}] \right) \quad (p_{t-1} \equiv \text{normalized } w_{t-1})
$$

$$
\leq (e^{-\eta} - 1) \mathbb{E}_{i \sim p_{t-1}} [L(\hat{y}_{t,i}, y_t)] \quad \text{(ineq. (1))}
$$

$$
\leq (e^{-\eta} - 1) L(\hat{y}_t, y_t). \quad \text{(Jensen’s ineq.)}
$$

Summing up these inequalities yields the upper bound

$$
\Phi_T - \Phi_0 \leq (e^{-\eta} - 1) \sum_{t=1}^{T} L(\hat{y}_t, y_t).
$$

We can also derive a lower bound straightforwardly:

$$
\Phi_T - \Phi_0 = \log \sum_{i=1}^{N} e^{-\eta L_{r,i}} - \log N \geq \log \max_{i \in [1, N]} e^{-\eta L_{r,i}} - \log N
$$

$$
= -\eta L^*_T - \log N.
$$

Comparing the lower and upper bound gives

$$
-\eta L^*_T - \log N \leq (e^{-\eta} - 1) \sum_{t=1}^{T} L(\hat{y}_t, y_t)
$$

$$
\Leftrightarrow \sum_{t=1}^{T} L(\hat{y}_t, y_t) \leq \frac{\eta L^*_T}{1 - e^{-\eta}} + \frac{\log N}{1 - e^{-\eta}}.
$$

For $L^*_T = 0$ and $\eta$ large, we retrieve the Halving bound. \hfill \Box

3. Bonus question: Prove the inequality $\frac{\eta}{1 - e^{-\eta}} \leq 1 + \eta$. Use this to derive an upper bound and choose $\eta$ to minimize that bound.

Solution:

$$
\frac{\eta}{1 - e^{-\eta}} \leq 1 + \eta
$$

$$
\Leftrightarrow \eta \leq 1 + \eta - \eta e^{-\eta} - e^{-\eta}
$$

$$
\Leftrightarrow 0 \leq 1 - \eta e^{-\eta} - e^{-\eta}
$$

$$
\Leftrightarrow 1 + \eta \leq e^{-\eta}
$$

2
and the last line is always true for \( \eta > 0 \).

Applying this inequality leads to the upper bound:

\[
(1 + \eta) L^* T + \frac{\log N}{1 - e^{-\eta}};
\]

and minimizing this expression in \( \eta \) leads to the following equation:

\[
L^* T - \frac{\log N}{(1 - e^{-\eta})^2} e^{-\eta} = 0,
\]

which is quadratic in \( e^{-\eta} \).

\[\square\]

C. Correlated equilibria

Consider the following version of the Rock-Paper-Scissors where players are both penalized if they play the same action.

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<thead>
<tr>
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<th>R</th>
<th>P</th>
<th>S</th>
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</thead>
<tbody>
<tr>
<td>R</td>
<td>(-1, -1)</td>
<td>(0, 1)</td>
<td>(1, 0)</td>
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<tr>
<td>P</td>
<td>(1, 0)</td>
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<tr>
<td>S</td>
<td>(0, 1)</td>
<td>(1, 0)</td>
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</tr>
</tbody>
</table>

1. Show that this game admits a unique mixed Nash equilibrium with non-zero probability for all actions. What is the expected payoff for the players?

*Solution:* Let \( c = (c_1, c_2, 1-c_1-c_2) \in \Delta_1 \) and \( r = (r_1, r_2, 1-r_1-r_2) \in \Delta_1 \) be column player and row player’s strategy respectively. Suppose \((c, r)\) is a mixed Nash equilibrium. First, assume \( c_1 > 0, c_2 > 0, 1-c_1-c_2 > 0 \). Then the expected payoff for column player to play R, P or S should be same, otherwise column player has incentive to remove probability from suboptimal action(s). This implies that

\[-r_1 + (1 - r_1 - r_2) = r_1 - r_2 = r_2 - (1 - r_1 - r_2),\]

which leads to a unique solution \( r = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). Since \( r_1 = r_2 = 1 - r_1 - r_2 > 0 \), by symmetry \( c = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). The expected payoff for both players are 0.

Now we prove this is the unique Nash equilibrium. Obviously there does not exist any pure Nash equilibrium. Without loss of generality, assume \( c_1 > 0, c_2 > 0, 1-c_1-c_2 = 0 \). By the same argument, we must have

\[-r_1 + (1 - r_1 - r_2) = r_1 - r_2,\]
which implies \( r_1 = \frac{1}{3} \). By symmetry, \( c_1 = \frac{1}{3} \) and therefore \( c = \left( \frac{1}{3}, \frac{2}{3}, 0 \right) \). Now the expected payoffs for row player to play R, P, S are \(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\) respectively. This however suggests row player should put all probability on action S, which contradicts with \( r_1 = \frac{1}{3} \). Therefore \( ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})) \) is the only mixed Nash equilibrium.

2. Show that the game admits a correlated equilibrium with expected payoff \( \frac{1}{2} \).

*Solution:* Consider the following strategy:

<table>
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<tr>
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<th>R</th>
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<th>S</th>
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</thead>
<tbody>
<tr>
<td>R</td>
<td>0</td>
<td>1/6</td>
<td>1/6</td>
</tr>
<tr>
<td>P</td>
<td>1/6</td>
<td>0</td>
<td>1/6</td>
</tr>
<tr>
<td>S</td>
<td>1/6</td>
<td>1/6</td>
<td>0</td>
</tr>
</tbody>
</table>

It is a correlated equilibrium. Consider the payoff for row player if he plays action R. We have:

\[
p(R, R)u(R, R) + p(R, P)u(R, P) + p(R, S)u(R, S) = \frac{1}{6}
\]

\[
p(R, R)u(P, R) + p(R, P)u(P, P) + p(R, S)u(P, S) = -\frac{1}{6} < \frac{1}{6}
\]

\[
p(R, R)u(S, R) + p(R, P)u(S, P) + p(R, S)u(S, S) = 0 < \frac{1}{6}
\]

By symmetry we can show these inequalities hold if row player play other actions. We have same results for column players as well. Therefore this is a correlated equilibrium. The expected payoff is \( \frac{1}{6} \times 3 = \frac{1}{2} \).

\(\square\)

D. Mirror Descent

In class we presented a general guarantee for Mirror Descent. We will adopt the assumptions of that theorem as well as the notation. Additionally, assume that the functions \( f_t \) are \( \beta \)-strongly convex with respect to \( \Phi \), that is

\[
f_t(w') \geq f_t(w) + \delta f_t(w) \cdot (w' - w) + \frac{\beta}{2} B(w' \parallel w),
\]

for all \( w', w \in \mathbb{R}^N \), with \( \beta > 0 \).
1. Suppose we use Mirror Descent with a time-varying learning rate $\eta_{t+1} = \frac{2}{t}$. Prove a logarithmic bound on the regret of Mirror Descent (hint: use proof for strongly convex losses given for PSGD). Your bound should be explicit and you should carefully justify all steps.

Solution:

$R_T(MD) = \sum_{t=1}^{T} (f_t(w_t) - f_t(w^*))$

$= \sum_{t=1}^{T} \delta f_t(w_t) \cdot (w_t - w^*) - \frac{\beta}{2} B(w^* \parallel w_t)$

$= \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [\nabla \Phi(w_t) - \nabla \Phi(v_{t+1})] \cdot (w_t - w^*) - \frac{\beta}{2} B(w^* \parallel w_t)$

$= \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [B(w^* \parallel w_t) - B(w^* \parallel v_{t+1}) + B(w_t \parallel v_{t+1})]$  (def. of $v_t$)

$= \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [B(w^* \parallel w_t) - B(w^* \parallel v_{t+1}) - B(w_t \parallel v_{t+1}) + B(w_t \parallel v_{t+1})]$  (Breg. div. Identity)

$\leq \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [B(w^* \parallel w_t) - B(w^* \parallel v_{t+1}) - B(w_t \parallel v_{t+1})]$  (Pythagorean ineq.)

$= \sum_{t=1}^{T} \frac{\beta}{2} [(t - 1)B(w^* \parallel w_t) - tB(w^* \parallel w_{t+1})]$

$+ \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [B(w_t \parallel v_{t+1}) - B(w_{t+1} \parallel v_{t+1})]$  (telescoping sum)

$= \frac{\beta}{2} \left[ -TB(w^* \parallel w_{T+1}) \right] + \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [B(w_t \parallel v_{t+1}) - B(w_{t+1} \parallel v_{t+1})]$  (telescoping sum)

$\leq \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [B(w_t \parallel v_{t+1}) - B(w_{t+1} \parallel v_{t+1})]$  (Breg. div. $\geq 0$)

Now, using the $\alpha$-strong convexity of $\Phi$ with respect to $\| \cdot \|$, we can
write

\[ B(w_t \| v_{t+1}) - B(w_{t+1} \| v_{t+1}) \]
\[ = \Phi(w_t) - \Phi(w_{t+1}) - \nabla \Phi(v_{t+1}) \cdot (w_t - w_{t+1}) \]
\[ \leq (\nabla \Phi(w_t) - \nabla \Phi(v_{t+1})) \cdot (w_t - w_{t+1}) - \frac{\alpha}{2} \|w_t - w_{t+1}\|^2 \quad (\alpha\text{-strong convexity}) \]
\[ = \eta_{t+1} \delta f_t(w_t) \cdot (w_t - w_{t+1}) - \frac{\alpha}{2} \|w_t - w_{t+1}\|^2 \quad \text{(def. of } v_{t+1}) \]
\[ \leq \eta_{t+1} G_s \|w_t - w_{t+1}\| - \frac{\alpha}{2} \|w_t - w_{t+1}\|^2 \quad (G_s\text{-Lipschitzness}) \]
\[ \leq \frac{(\eta_{t+1} G_s)^2}{2\alpha}. \quad \text{(max. of 2nd deg. eq.)} \]

Thus,

\[ R_T(\text{MD}) \leq \frac{G_s^2}{2\alpha} \sum_{t=1}^{T} \eta_{t+1} = \frac{G_s^2}{2\alpha} \sum_{t=1}^{T} \frac{2}{\beta t} \leq \frac{G_s^2}{\alpha \beta} (1 + \log T). \]

\[ \square \]

2. Show that your bound coincides with the one presented for PSGD in class in the case of strongly-convex losses.

\textit{Solution:} PSGD coincides with Mirror Descent for the choice of the potential \( \Phi(w) = \|w\|^2 \) which is 2-strongly convex (\( \alpha = 2 \)). \( \square \)

\section*{B. Time-varying parameter}

In this problem, we consider the use of the EW algorithm with no prior knowledge of the horizon \( T \) and using, instead of a fixed parameter \( \eta \), a time-varying parameter \( \eta_t > 0 \), with \( \eta_t \leq \eta_{t-1} \) for all \( t \in [1, T] \). We define \( w_{0,i} = 1 \) for all \( i \in [1, N] \). At iteration \( t \geq 1 \), prediction is made using the weights \( w_{t-1,i} \) via

\[ \hat{y}_t = \frac{\sum_{i=1}^{N} w_{t-1,i} y_{t,i}}{\sum_{i=1}^{N} w_{t-1,i}}. \quad (3) \]

The weight of expert \( i \) is then updated as follows:

\[ w_{t,i} = e^{-\eta_t L_{t,i}} \quad \text{with} \quad L_{t,i} = \sum_{s=1}^{t} L(\hat{y}_{s,i}, y_s). \quad (4) \]
For any $t$, we also define $L_{t,*} = \min_{i \in [1,N]} L_{t,i}$ and $w_{t,*} = e^{-\eta_t L_{t,*}}$. We define $W_t$ as the sum of the weights at time $t$: $W_t = \sum_{i=1}^N w_{t,i}$. Similarly, we define $w'_{t,i} = e^{-\eta_{t-1} L_{t,i}}$, $W'_t = \sum_{i=1}^N w'_{t,i}$, and $w'_{t,*} = e^{-\eta_{t-1} L_{t,*}}$.

For any $t \in [0,T]$, define the potential

$$\Phi_t = \frac{1}{\eta_t} \log \frac{W_t}{w_{t,*}}.$$  

(5)

1. Show that the following equality holds for all $t \in [1,T]$:

$$\Phi_t - \Phi_{t-1} = \frac{1}{\eta_t} \left[ \log \frac{W_t}{w_{t,*}} - \log \frac{W'_{t-1}}{w'_{t-1,*}} \right] + \frac{1}{\eta_t - \eta_{t-1}} \log \frac{W'_t}{w'_{t,*}} + \frac{1}{\eta_{t-1}} \left[ \log \frac{W'_{t-1}}{w'_{t-1,*}} - \log \frac{W_{t-1}}{w_{t-1,*}} \right].$$  

(6)

**Solution:** By definition of the potential, we can write by adding and subtracting the same terms

$$\Phi_t - \Phi_{t-1} = \frac{1}{\eta_t} \log \frac{W_t}{w_{t,*}} - \frac{1}{\eta_{t-1}} \log \frac{W_{t-1}}{w_{t-1,*}}$$

$$= \frac{1}{\eta_t} \left[ \log \frac{W_t}{w_{t,*}} - \log \frac{W'_{t-1}}{w'_{t-1,*}} \right] + \frac{1}{\eta_t - \eta_{t-1}} \log \frac{W'_t}{w'_{t,*}} + \frac{1}{\eta_{t-1}} \left[ \log \frac{W'_{t-1}}{w'_{t-1,*}} - \log \frac{W_{t-1}}{w_{t-1,*}} \right].$$

□

2. Use the inequality:

$$\log \frac{\sum_{i=1}^N e^{-\eta_t [L_{t,i} - L_{t,*}]}}{\sum_{i=1}^N e^{-\eta_{t-1} [L_{t,i} - L_{t,*}]}} \leq \frac{\eta_{t-1} - \eta_t}{\eta_{t-1}} \log N,$$  

(7)

to show the following bound on $A$:

$$A \leq \left[ \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right] \log N.$$  

(8)

Bonus question: prove inequality (7) using Jensen’s inequality and the convexity of the function $\Psi: \eta \mapsto \log \left[ \sum_{i=1}^n e^{-\eta [L_{t,i} - L_{t,*}]} \right]$.  

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Solution: We first show inequality (7). By Jensen’s inequality
\[
\log \frac{\sum_{i=1}^{N} e^{-\eta_i [L_{t,i} - L_{t,*}]} }{\sum_{i=1}^{N} e^{-\eta_* [L_{t,i} - L_{t,*}]} } \\
= - \log \frac{\sum_{i=1}^{N} e^{-\eta_* [L_{t,i} - L_{t,*}]} }{\sum_{i=1}^{N} e^{-\eta_i [L_{t,i} - L_{t,*}]} } \\
= - \log \frac{\sum_{i=1}^{N} e^{-\eta_* [L_{t,i} - L_{t,*}]} e^{[\eta_* - \eta_i] [L_{t,i} - L_{t,*}]} }{\sum_{i=1}^{N} e^{-\eta_i [L_{t,i} - L_{t,*}]} } \\
= - \log \left[ e^{[\eta_* - \eta_i] [L_{t,i} - L_{t,*}]} \right] \\
\leq \mathbb{E} \left[ e^{[\eta_* - \eta_i] [L_{t,i} - L_{t,*}]} \right] \\
= (\eta_* - \eta_i) \mathbb{E} [L_{t,i} - L_{t,*}] \\
= -(\eta_* - \eta_i) \Psi'(\eta_i),
\]
where \( \Psi(\eta) = \log \left[ \sum_{i=1}^{N} e^{-\eta_i [L_{t,i} - L_{t,*}]} \right] \). Since \( \Psi \) is convex, the following inequality holds:
\[
\Psi(0) - \Psi(\eta_{t-1}) \geq \Psi'(\eta_{t-1})(0 - \eta_{t-1}) = -\eta_{t-1} \Psi'(\eta_{t-1}).
\]
Observe that \( \Psi(0) = \log N \) and that \( \Psi(\eta_{t-1}) \geq 0 \) (since \( L_{t,i} - L_{t,*} = 0 \) for some \( i \)). This implies \( -\Psi'(\eta_{t-1}) \leq \frac{\Psi(0)}{\eta_{t-1}} = \frac{\log N}{\eta_{t-1}} \) and
\[
\log \frac{\sum_{i=1}^{N} e^{-\eta_i [L_{t,i} - L_{t,*}]} }{\sum_{i=1}^{N} e^{-\eta_* [L_{t,i} - L_{t,*}]} } \leq \frac{\eta_{t-1} - \eta_t}{\eta_{t-1}} \log N.
\]
Now, \( A \) can be written as follows and bounded using that inequality:
\[
A = \frac{1}{\eta_t} \log \frac{\sum_{i=1}^{N} e^{-\eta_i [L_{t,i} - L_{t,*}]} }{\sum_{i=1}^{N} e^{-\eta_* [L_{t,i} - L_{t,*}]} } \\
\leq \frac{1}{\eta_t} \frac{\eta_{t-1} - \eta_t}{\eta_{t-1}} \log N \\
= \left[ \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right] \log N.
\]
\[
\square
\]
3. Show that \( B \) can be bounded as follows:
\[
B \leq \left[ \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right] \log N. \tag{9}
\]
Solution: By definition of $W_t'$,

$$\log \frac{W_t'}{w_{t,*}} = \log \sum_{i=1}^{N} \frac{w_{t,i}'}{w_{t,*}} \leq \log N,$$

since $\frac{w_{t,i}'}{w_{t,*}} \leq 1$ for all $i \in [1, N]$. Since $\eta_t \leq \eta_{t-1}$, this implies $B \leq \left[ \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right] \log N$. \hfill \Box

4. Use a technique similar to that of the proof of EW given in class to show that the third term can be bounded as follows:

$$C \leq L_{t,*} - L_{t-1,*} - L(\hat{y}_t, y_t) + \frac{\eta_{t-1}}{8}. \quad (10)$$

Solution: $C$ can be rewritten as

$$C = \frac{1}{\eta_{t-1}} \log \frac{\sum_{i=1}^{N} e^{-\eta_{t-1}[L_{t,i} - L_{t-1,i}]} }{\sum_{i=1}^{N} e^{-\eta_{t-1}[L_{t-1,i} - L_{t-1-1,i}]} }$$

$$= \frac{1}{\eta_{t-1}} \left[ \log e^{\eta_{t-1}[L_{t,*} - L_{t-1,*}]} + \log \frac{\sum_{i=1}^{N} e^{-\eta_{t-1}L_{t,i}} }{\sum_{i=1}^{N} e^{-\eta_{t-1}L_{t-1,i}} } \right]$$

$$= L_{t,*} - L_{t-1,*} + \frac{1}{\eta_{t-1}} \log \frac{\sum_{i=1}^{N} e^{-\eta_{t-1}L_{t-1,i}} e^{-\eta_{t-1}L(\hat{y}_t, y_t)} }{\sum_{i=1}^{N} e^{-\eta_{t-1}L_{t-1,i}} }.$$

Using Hoeffding's inequality to bound the second term, we obtain:

$$C \leq L_{t,*} - L_{t-1,*} - L(\hat{y}_t, y_t) + \frac{\eta_{t-1}}{8}. \quad (10)$$

\hfill \Box

5. Use the upper bounds on $A$, $B$, and $C$ to prove the following upper bound on the regret of EW using a time-varying parameter:

$$R_T \leq \frac{1}{8} \sum_{t=1}^{T} \eta_{t-1} + \frac{2}{\eta_T} \log N - \frac{1}{\eta_0} \log N. \quad (11)$$

Solution: By the bounds on $A$, $B$, and $C$,

$$\Phi_t - \Phi_{t-1} \leq L_{t,*} - L_{t-1,*} - L(\hat{y}_t, y_t) + \frac{\eta_{t-1}}{8} + 2 \left[ \frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right] \log N,$$
Thus, summing over $t$ yields

$$\Phi_T - \Phi_0 \leq L_{T,*} - \sum_{t=1}^T L(\hat{y}_t, y_t) + \frac{1}{8} \sum_{t=1}^T \eta_{t-1} + \frac{1}{\eta_T} - \frac{1}{\eta_0} \log N.$$  

The following straightforward lower bound also holds:

$$\Phi_T - \Phi_0 = \frac{1}{\eta_T} \log \frac{W_T}{w_{T,*}} - \frac{1}{\eta_0} \log N \geq - \frac{1}{\eta_0} \log N,$$

since $W_T \geq w_{T,*}$. Comparing the upper and lower bounds and rearranging terms yields the desired result.

6. Assume now that for any $t \in [0, T]$, we choose $\eta_t = \sqrt{\frac{\alpha \log N}{t+1}}$, where $\alpha > 0$ is a parameter we will select. Show that the following upper bound holds:

$$R_T \leq \frac{1}{4} \sqrt{\alpha T \log N} + 2 \sqrt{(T + 1) \log N} \frac{\log N}{\alpha} - \sqrt{\log N \frac{\log N}{\alpha}}.$$

Solution: In view of the following bound on the sum starting from 2

$$\sum_{t=2}^T \frac{1}{\sqrt{t}} \leq \sum_{t=2}^T \int_{t-1}^t \frac{du}{\sqrt{u}} = \int_2^T \frac{du}{\sqrt{u}} = 2 \left[ \sqrt{T} - 1 \right],$$

we can write $\sum_{t=1}^T \frac{1}{\sqrt{t}} \leq 2\sqrt{T}$, and thus $\sum_{t=1}^T \eta_{t-1} \leq 2\sqrt{\alpha T \log N}$.  

7. Prove the following regret bound using $\alpha = 8$:

$$R_T \leq \sqrt{2T \log N} + \sqrt{\log N \frac{\log N}{8}}.$$  

Solution: Using the sub-additivity of the square-root function, we can first simplify the inequality of the previous question:

$$R_T \leq \frac{1}{4} \sqrt{\alpha T \log N} + 2 \sqrt{(T + 1) \log N} \frac{\log N}{\alpha} - \sqrt{\log N \frac{\log N}{\alpha}}$$

$$\leq \frac{1}{4} \sqrt{\alpha T \log N} + 2 \sqrt{T \log N \frac{\log N}{\alpha}} + 2 \sqrt{\log N \frac{\log N}{\alpha}} - \sqrt{\log N \frac{\log N}{\alpha}}$$

$$= \frac{1}{4} \sqrt{\alpha T \log N} + 2 \sqrt{T \log N \frac{\log N}{\alpha}} + \sqrt{\log N \frac{\log N}{\alpha}}.$$
Next, we choose $\alpha$ to minimize the sum of the first two terms, which is obtained by making these terms be equal. That gives $\alpha = 8$. Plugging in that value of $\alpha$ in the inequality above yields immediately the inequality.

8. Bonus question: Derive a regret bound in terms of $L_{T,*}$ by choosing $\eta_t = \sqrt{\frac{\alpha \log N}{L_{t,*}^2 + 1}}$.  

$\square$