

April 7 2015

**LECTURE # 8**

GAME THEORY

Study of strategic interactions ...

◊ Choices

◊ Rational Decisions

Strategic Choices.

◊ Static Games

◊ Dynamic Games

◊ Evolutionary Games.

{ Signalling  
Bargaining

Issues: ◊ Privacy (Information Asymmetry)

◊ Trust (Correlation of Encounter)

- Anonymity

- Liquidity

◊ Signaling

◊ Coordination

- Bargaining

- Pricing

- Auctioning.

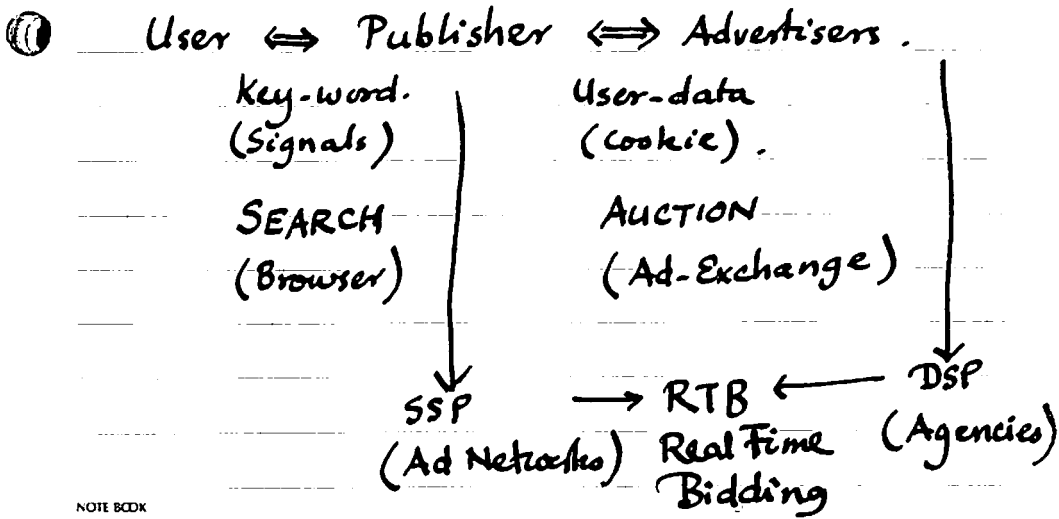
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		Advertisers	
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User	Click	(0, 0)	(0, -1)
	Click & Buy	(-1, 0)	(-1, -1)
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Note that in the absence of any information on the user, the advertiser is better off by not showing any ad.

⇒ Solution: User Information { Cookie  
Key-word  
3<sup>rd</sup> party data.

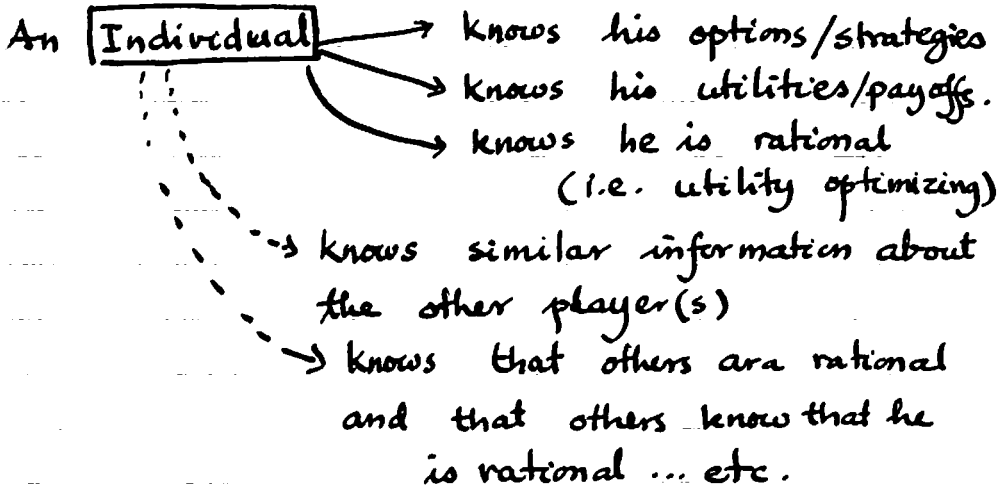
You need two games (Signaling Games).



## GAME THEORY.

KEY ASSUMPTIONS (often violated)

- 1) Rationality. (or Bounded Rationality)
- 2) CKR: Common Knowledge of Rationality



He strategically selects an option that optimizes his payoff, (while anticipating how others will behave.)

Note: ○ Payoff need not be just monetary  
 ○ Rationality should be treated as an idealization:

- Learning
- Statistical Inference / Data Science / Computation
- Evolutionary Stable Strategies.

## Strategic Form Games.

Defn: A strategic form game is a triplet.  
 $\langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle$

such that:

a) Index Set:  $I \equiv$  Finite set of players.  
 $\equiv \{1, 2, 3, \dots, l\}$

b) Strategy Set:  $S_i, i \in I \equiv$  Set of available actions for player  $i \in I$ .

c) Strategy Profile:  

$$S = \prod_i S_i$$

d) Utility Function:  

$$u_i: S \rightarrow \mathbb{R}$$

$\equiv$  The payoff function of player  $i \in I$ .

Notation  $S \equiv \prod_i S_i; s_i \in S_i$

$S_{-i} = \prod_{j \neq i} S_j \equiv$  Strategy Profile for all players except  $i \in I$ .

$\therefore S = S_i \times S_{-i}; s_{-i} \in S_{-i}$

$s_{-i} = \langle s_j \rangle_{j \neq i} \equiv$  Vector of all actions for all players excluding player  $i \in I$ .

$\langle s_i, s_{-i} \rangle =$  Strategy Profile.

Example: BOS (Battle of the Sexes)  
 $\equiv$  Two Player Game.

$I = \{F, M\}$  Two players of opposite sex  
 M = Male; F = Female.

$S_F = S_M = \{O, F\}$  O = Opera  
 F = Football.

$S = S_F \times S_M = \{ \langle O, O \rangle, \langle O, F \rangle, \langle F, O \rangle, \langle F, F \rangle \}$

F = Row Player M = Column Player

	M	
F	O	F
O	3, 2	0, 0
F	0, 0	2, 3

$U_F: S \rightarrow \mathbb{R}$

$U_M: S \rightarrow \mathbb{R}$

$$U_F(\langle O, O \rangle) = U_M(\langle F, F \rangle) = 3$$

$$U_F(\langle F, F \rangle) = U_M(\langle O, O \rangle) = 2$$

$\left\{ \begin{array}{l} \text{otherwise} \\ 0. \end{array} \right.$

## PLAY OF THE GAME

Each player chooses a strategy  $s_i$

↓  
Strategy Profile =  $\langle s_1, s_2, \dots, s_n \rangle \equiv s$

↓  
Utility =  $u_i(s)$

○ If  $s^* = \langle s_1^*, s_2^*, \dots, s_n^* \rangle = \text{"BEST"}$   
then

$$\forall i \in I \quad \forall s_i \in S_i \\ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

↓  
**Best Response**

① STABILITY: No player can profitably deviate given the strategy of the other players.

○ ② FIXED POINT UNDER CKR: Each player chooses a strategy  $(s_i^*)$  expecting all other players to choose "rationally."

# NASH EQUILIBRIUM

(PURE STRATEGY N.E.)

A pure strategy Nash Equilibrium of a strategic game:

$$\langle I, (s_i)_{i \in I}, (u_i)_{i \in I} \rangle$$

is a strategy profile  $s^* \in S$  such that

$$\forall i \in I \quad \forall s_i \in S_i: u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

Example BoS:

◊ GREEDY STRATEGY  $\not\Rightarrow$  NE.

$$S_F = \text{opera}, \quad S_M = \text{football} \Rightarrow \text{payoff} = (0, 0)$$

Either  $S_F$  should deviate to football:

$$\text{payoff } 0 \rightarrow 2.$$

OR  $S_M$  should deviate to opera

$$\text{payoff } 0 \rightarrow 2.$$

◊ ULTRA ALTRUISTIC STRATEGY  $\not\Rightarrow$  NE.

$$S_F = \text{football}, \quad S_M = \text{opera} \Rightarrow \text{payoff} = (0, 0)$$

Either  $S_F$  should deviate to opera

$$\text{payoff } 0 \rightarrow 3$$

OR  $S_M$  should deviate to football

$$\text{payoff } 0 \rightarrow 3$$

There exist

TWO NASH EQUILIBRIA (Pure Strategy)

$\langle 0, 0 \rangle$  or  $\langle F, F \rangle$

$\swarrow$   $\langle \text{opera, opera} \rangle$

$\langle \text{football, foot ball} \rangle$

F enjoys both the opera & M's company  
M gains utility by being in F's company.

Example. Beautiful Mind.

	JNF	Blonde	Brunette
JN			
Blonde		(0, 0)	(2, 1)
Brunette		(1, 2)	(1/2, 1/2)

In the movie JNF selects brunette  
 $\Rightarrow$  Not Rational!  
Where is the Nash Equilibrium?



Example (Extended) Prisoner's Dilemma (P.D.)

		P <sub>2</sub>		
		Confess	Silence	suicide
P <sub>1</sub>	Confess	(-2, -2)	(0, -3)	(-2, -10)
	Silence	(-3, 0)	(-1, -1)	(0, -10)
	Suicide	(-10, -2)	(-10, 0)	(-10, -10)

- 1) Suicide is dominated  
⇒ Eliminate.
- 2) silence is dominated  
⇒ Eliminate.

↓  
 <Confess, Confess>  
 ≡ P.S. N.E.

Rock. Paper-Scissors.

∄ P.S. N.E.

		P <sub>2</sub>		
		R	P	S
P <sub>1</sub>	R	(0, 0)	(-1, 1)	(1, -1)
	P	(1, -1)	(0, 0)	(-1, 1)
	S	(-1, 1)	(1, -1)	(0, 0)

ZERO-SUM GAME.

↓  
MIXED STRATEGY.

## MIXED-STRATEGY

$\Sigma_i =$  Prob. Measure over Pure Strategies  $S_i$

$$\sigma_i = (p_{i1}, p_{i2}, \dots, p_{ik})$$

$$p_{ij} = \Pr [S_{ij} \in S_i \text{ is played}]$$

$$p_{ij} \geq 0 \quad \sum p_{ij} = 1.$$

$\Sigma = \prod_i \Sigma_i \cong$  Mixed Strategy Profile.

$$\sigma \in \Sigma$$

$$u_i(\sigma) = \int_S u_i(s) d\sigma(s) = \sum p_{ij} u_i(s_{ij}; \sigma_{-i})$$

### MIXED STRATEGY NASH EQUILIBRIUM:

$$\sigma^* \in \Sigma \cong \text{M.S.N.E.}$$

$$\forall i \quad \forall \sigma_i \in \Sigma_i \quad \overset{\text{iff}}{u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)}$$

NASH'S THM ( $\because$  Kakutani Fixed Point Thm.)

Every finite game has a

Mixed Strategy Nash Equilibrium.