$G = (V, E)$: Undirected / Directed } graph.

Connected Component:

A connected component of a graph is defined as a maximal subgraph in which path exists from every node to every other.

◇ A path in a graph is closed if its start and end vertices coincide:

$$v_0, v_1, \dots, v_n \equiv v_0$$
$$\forall_i \ (v_i, v_{i+1}) \in E \qquad 0 \le i < n$$
$$\forall_{i,j} \ v_i \ne v_j \qquad 0 \le i < j \le n$$

◇ A cycle is defined as a closed path in which $n \ge 3$.

Strongly Connected Component

A strongly connected component of a graph is defined as a maximal subgraph in which cycle exists connecting every node to every other.

TREE : A tree is a connected graph that contains no cycle.

ADJACENCY MATRIX. (Undir)

Every graph $G = (V, E)$ with $|V| = n$ has associated with it a symmetric adjacency matrix $A \in \{0,1\}^{n \times n}$

Binary $n \times n$ matrix A in which

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \ \& \ v_j \text{ are adjacent } (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since in an undirected graph

$$(v_i, v_j) \equiv (v_j, v_i) \qquad a_{ij} = a_{ji}$$

$$A^T = A. \leftarrow \text{A real-valued symmetric matrix.}$$

$$d_{ii} = d(v_i) = |\{ v_j \mid (v_i, v_j) \in E \}| \text{ : Degree.}$$

$$D = \begin{bmatrix} d_{11} & & 0 \\ & d_{22} & \\ 0 & & \ddots \ d_{nn} \end{bmatrix} \text{ : Diagonal Matrix.}$$

$$\text{Trace } D = \sum_{v_i} d(v_i) = 2m.$$

Boundary Matrix $\mathcal{B}$. $\in \{-1,0,+1\}^{m\times n}$

Columns are indexed by the vertices of G.
Rows are indexed by the edges of G

$$\mathcal{B}(e,v) = \begin{cases} +1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{B}^T B = L = D - A.$

Choose edge directions arbitrarily if G = undir.

$L = \mathcal{D} - A.$

Random Surfing on G.

$$\mathcal{P}(u,v) = \begin{cases} \frac{1}{d_v} & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{P} = \mathcal{D}^{-1} A.$

$L = \mathcal{D}(I - P) = \mathcal{D}\Delta; \quad \Delta \equiv I - P$

$\Delta = I - P \equiv$ (Discrete) Laplace operator.

Your position within the social network assigns a social-value.

$\Rightarrow$ Rank

(e.g. page rank).

$f(x)$

$f: V \to \mathbb{R}$ a scalar rank function.

Dirichlet Sum of G

$$\sum_{(u,v)\in E} (f(u) - f(v))^2$$

You'd like to choose ranks so that the sum is minimized:

Avoid the trivial rank: $f(x) = 1 \ \forall x$.

$\Rightarrow$ Focusing on the relative values.

$$\Delta f(x) = \frac{1}{d_x} \sum_{(y,x)\in E} (f(x) - f(y))$$

$$\Downarrow$$

$$(I - P) f.$$

RANDOM SURFER MODEL.

Imagine a web surfer bouncing along randomly following the graph (hyperlink graph of the web.)

When the surfer arrives at a node he chooses at random hyper-links (directed edge) to a new node.

Asymptotically, the propotion of time the random surfer spends on a given node/page is a measure of

$\boxed{\text{Relevance (Relative Importance)}}$

Dangling Nodes - Sinks
Periodicity in the graph
} Problems.

Stochastic Teleportation.

NOTE BOOK