# LECTURE #11 December 4 2014. 

## UNIQUE GAMES PROBLEM.

CUT:

Given a graph $G = (V, E)$, a cut is a partion of the vertices $V = S \cup T$ ($S \cap T = \phi$) into disjoint sets $S$ and $T$. The cut, $C$, consist of edges that cross from $S$ to $T$.

$$C = \{\langle u, v\rangle \mid \langle u,v\rangle \in E,\ u \in S,\ v \in T\} \subseteq E$$

Size of the cut is $|C|$.

MIN CUT: $\min\limits_{S \subseteq V} \left|\{\langle u,v\rangle \in E \mid u \in S,\ v \in V \setminus S\}\right|$

st-MIN CUT: $\min\limits_{\substack{S \subseteq V \\ s \in S \\ t \in V\setminus S}} \left|\{\langle u,v\rangle \in E \mid u \in S,\ v \in V \setminus S\}\right|$

st-MINCUT $\in P$ $\left\{\begin{array}{l}\text{Can be solved using} \\ \text{a polynomial time algorithm} \\ \text{for MAXFLOW \& duality.}\end{array}\right.$

$\Rightarrow$ MINCUT $\in P$.

MAXCUT: $\max\limits_{S \subseteq V} \left|\{\langle u,v\rangle \in E \mid u \in S,\ v \in V \setminus S\}\right|$.

MAXCUT = NP complete.

MAX CUT.

Data: A graph G and an integer k.

Desiderata: Does there exist a cut in G of size k or greater?

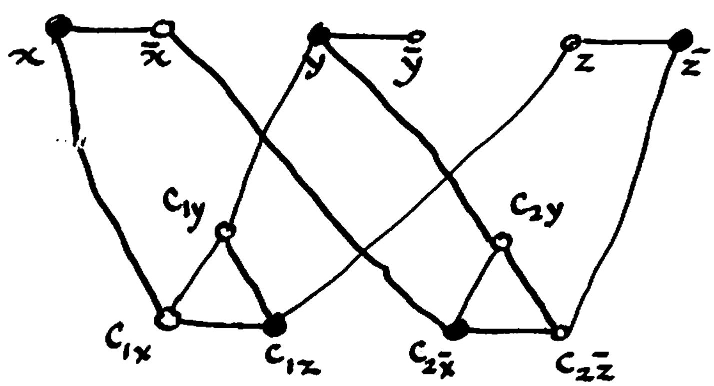
NAE-3-SAT $\leq$ MAX CUT.

$\varphi$ = 3-CNF made up of clauses $C_1, \dots, C_m$ each clause consisting of literals $C_i = \{l_{i1}, l_{i2}, l_{i3}\}$ over the variables $x_1, \dots, x_n$.

E.g. $(x \vee y \vee z) \wedge (\bar{x} \vee y \vee \bar{z})$

Create a graph G with $2n + 3m$ vertices and ~~$3n+6m$~~ $n + 6m$ edges.

$k = n + 5m$.




Assignment

$x = 1$

$y = 1$

$z = 0$

All edges except $\langle c_{1x}, c_{1y}\rangle$, and $\langle c_{2y}, c_{2\bar{z}}\rangle$ are included in the cut

$k = 3 + 5 \times 2 = 13$

is satisfied.

## Constraint Solving

Describe each edge by a constraint

$x_i \neq x_j$ for every edge $\langle i,j\rangle \in E$

$x_i, x_j \in \{-1, 1\}$

$$\boxed{\max \ \frac{1}{2} \sum_{\langle i,j\rangle \in E} 1 - x_i x_j \ , \quad \text{s.t. } x_i, x_j \in \{-1, 1\}}$$

More generally,

$$x_i + x_j \equiv 1 \mod 2 \quad \forall \langle i,j \rangle \in E$$

$$x_i \in \{0,1\} \in \mathbb{Z}_2.$$

Finite Field.

Satisfy as many equations of this kind as possible.

<u>Generalization</u>: Unique Game.

Maximum fraction of linear "binomial" equations that can be satisfied by any assignment.

$$x_1 + x_3 \equiv 2 \mod k$$
$$3x_5 + x_2 \equiv -1 \mod k$$
$$x_2 + 5x_1 \equiv 0 \mod k$$

The system of linear equations is $(1-\epsilon)$-sound if it has a sub-system of ~~is~~ equations whose size is $(1-\epsilon)$ fraction of the entire system and it is solvable.

The system of linear equations is $\delta$-~~sound~~ complete if it ~~has~~ is possible to find in polynomial time a subsystem of equations whose size is $\delta$-fraction of the entire system and it is solvable.

Reformulating it as a graph problem: 80

For each variable introduce a node $x_i$
For each equation introduce an edge an edge $e_{ij}$, if the equation involves variables $x_i$ and $x_j$
For each edge introduce the unique constraint by a permutation over k-colors $\pi_e \in S_k$

$$\mathcal{U}\left( G=(V,E), [k], \{\pi_e \mid e \in E\} \right)$$

MAX CUT $\Rightarrow$ $\mathcal{U}\left( G=(V,E), \{0,1\}, \left\{ \pi_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,\middle|\, e \in E \right\} \right)$

UNIQUE GAME $\quad \mathcal{U}(G=(V,E), [k], \{\pi_e \mid e \in E\})$
$\equiv$ Constraint Satisfaction Problem.

$G=(V,E)$ = Directed Graph.
$[k]$ = Colors.
$\pi_e \in S_k$ = A constraint represented by a bijection $\pi_e : [k] \to [k]$.

Compute a labelling $\mathcal{L}: V \to [k]$.

$e=(u,v)$ $\quad \mathcal{L} \models \pi_e$ (satisfies) $\quad$ iff $\pi_e(\mathcal{L}(u)) = \mathcal{L}(v)$

$$\mathrm{Opt}(\mathcal{U}) := \max_{\mathcal{L}: V \to [k]} \frac{1}{|E|} \left| \{ e \in E \mid \mathcal{L} \models \pi_e \} \right|$$

Data: A $\mathcal{U}(G=(V,E), [k], \{\pi_e \mid e \in E\})$ with $\mathrm{Opt}(\mathcal{U}) \geq 1-\epsilon$ ($\epsilon$ = fixed but not revealed)

Desiderata Compute $\mathrm{Opt}(\mathcal{U})$.

# Unique Games Conjecture: (Khot 2002) 

For every $\epsilon, \delta > 0$ there exists a constant $k = k(\epsilon, \delta)$ such that given a unique game instance

$$\mathcal{U}\left(G = (V,E), [k], \{\pi_e \mid e \in E\}\right)$$ it is

NP hard to distinguish between the two cases:

$(1-\epsilon)$-Soundness: $\mathrm{OPT}(\mathcal{U}) \geq 1-\epsilon$

$\delta$-Incompleteness: $\mathrm{OPT}(\mathcal{U}) \leq \delta$

- Unique Game = Maximization problem that is hard to approximate.
- Gap Preserving ~~Approximation~~ Reduction — Vertex-Cover, Max Cut, Graph Coloring, Systems of Equations.
- 2 Prover 1-round Game.

  2 provers coordinate to color the vertices (but do not cooperate subsequently)
  1 Verifier queries the two provers independently about the colors of the vertices $u$ and $v$ for a randomly selected edge $\langle u, v\rangle$.

  If UGC = false $\Rightarrow$ Pr (Provers win) $\geq 1-\epsilon$

  If UGC = True $\Rightarrow$ Pr (Verifier wins) $\geq \delta$

- PCP (Probabilistically Checkable Proofs)

  PCP Theorem: $\mathrm{PCP}(r(n), q(n))$ — $r(n)$: Randomness; $q(n)$: Query complexity

  $$\mathrm{PCP}(\log n, 1) = \mathrm{NP}.$$

$f : V \to \{-1, 1\}$

SDP- Relaxation :

$$\text{maximize} \sum_{\langle i,j\rangle \in E} \left(\frac{1 - v_i v_j}{2}\right).$$

$$\text{subject to} \quad v_i^2 = 1$$
$$v_i \in \{-1, 1\}$$

$$\Downarrow$$

$$\text{maximize} \sum_{\langle i,j\rangle \in E} \frac{1 - v_i \cdot v_j}{2}$$

$$\text{subject to} \quad \|v_i\|^2 = v_i \cdot v_i = 1$$
$$v_i \in \mathbb{R}^n$$

$$\Downarrow$$

$$\text{maximize} \quad L \cdot X$$

$L =$ Laplacian of the graph

$X = vv^T$

$$\text{subject to} \quad X_{ii} = 1 \quad \forall i$$

$$X \succeq 0$$

$$\Downarrow$$

This problem can be solved in polynomial time

$\rightarrow$ $X^* \succeq 0$

$X^* = V^T V$ be its cholesky factorization.

$\rightarrow$ $v_i = \dfrac{V_i}{\|V_i\|}$ ith column of V.

$\rightarrow$ Rounding Procedure

Choose uniformly a random vector $r$ on the unit n-sphere

$$\begin{cases} v_i^T r > 0 \longrightarrow \hat{v}_i = +1 \\ v_i^T r < 0 \longrightarrow \hat{v}_i = -1 \end{cases}$$

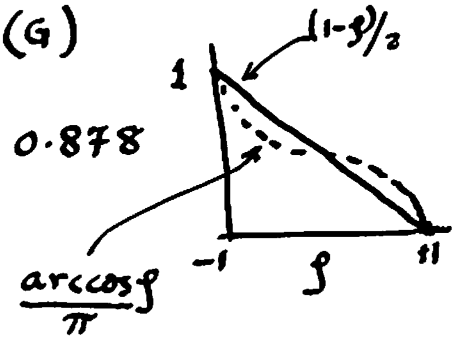
$$\Pr[(i,j) \in \text{Cut}] = \frac{\angle(\vec{v}_i, \vec{v}_j)}{\pi} = \frac{\arccos(\vec{v}_i \cdot \vec{v}_j)}{\pi}$$

$$\text{ALG}(G) = \sum_{\langle i,j\rangle \in E} \frac{\arccos(\vec{v}_i \cdot \vec{v}_j)}{\pi}$$

$$\text{SDP}(G) = \sum_{\langle i,j\rangle \in E} \frac{1 - \vec{v}_i \cdot \vec{v}_j}{2} \geq \text{OPT}(G)$$

$$\text{ALG}(G) \geq \alpha\, \text{SDP}(G) \geq \alpha\, \text{OPT}(G)$$

$$\alpha = \min_{-1 \leq \rho \leq 1} \frac{\arccos \rho / \pi}{(1-\rho)/2} \approx 0.878$$




## UNIQUE GAMES:

maximize $\displaystyle\sum_{e = \langle u,v\rangle \in E} \sum_{i \in [k]} \vec{x}_{u,i} \cdot \vec{x}_{v,\pi_e(i)}$

subject to

$$\forall_{v \in V} \quad \sum_{i \in [k]} \vec{x}_{v,i} \cdot \vec{x}_{v,i} = 1$$

$$\forall_{v \in V} \quad \forall_{\substack{i \neq j \\ i,j \in [k]}} \quad \vec{x}_{v,i} \cdot \vec{x}_{v,j} = 0$$

$$\forall_{u,v \in V} \quad \forall_{i,j \in [k]} \quad \vec{x}_{u,i} \cdot \vec{x}_{v,j} \geq 0$$