Lecture #4
February 19 2013.

pp1

Graph Theory:
a) Combinatorial Structures.
b) Spectral Properties.
c) Random Graphs and their Evolution.

Pair-wise Interactions: → Game Theory. (Strategic choices)
→ Graph Theory (Interaction choices)

Actors, Players $\longrightarrow V$
Links, Connection $\longrightarrow E \subseteq V \times V$ (Symmetric Binary Relation)

In addition, we will need Strategy Spaces S_v
& Pay-off functions:

$$u_v : \prod_{v \in V} S_v \to \mathbb{R}_+$$

Graphs: Defn
A graph - usually denoted $G(V,E)$ or $G = (V, E)$ - consists of a set of vertices V together with a set of edges $E \subseteq V \times V$.

· A graph is a mathematical object describing an (irreflexive, symmetric) binary relation on a discrete set (not necessarily finite)

Friendship:

(1) Irreflexive: One is not his own friend.
(2) Symmetric: One is a friend to a friend.
(3) NonTransitive: A friend's friend is not necessarily a friend.

Friendship is described by an (undirected) edge in the graph.

◇ An edge $e = (u,v) \in E$ (where $E \subseteq V \times V$) is defined by the ~~and~~ unordered pair of vertices (eg players) that serve as its end points.

◇ Two vertices u and v are adjacent if there exists an edge $e = (u,v)$ connecting them.

◇ The number of vertices is usually denoted by

$$|V| = n \quad \&$$

number of edges by

$$|E| = m.$$

$$m \le \frac{n(n-1)}{2} = \binom{n}{2}$$

n = # of ways to choose u
$n-1$ = # of ways to choose v
identify edge $(u,v) = (v,u)$.

Synonyms:

Vertices = {Nodes, Points} = {Actors, Agents, Players}

Edges = {Connections, Lines} = {Links, Ties, Encounters}

STRICT GRAPHS:

No Self-Loop $(u,u) \notin E$

No Multi-Edges $e_1 = (u_1, v_1) \neq e_2 = (u_2, v_2)$

$\Rightarrow (u_1 = u_2) \Rightarrow (v_1 \neq v_2)$

$\wedge (u_1 = v_2) \Rightarrow (v_1 \neq u_2)$

Two distinct social relationships that are empirically measured on the same group of people are represented by separate graphs.

Two vertices are <u>adjacent</u> if there is an edge connecting them.

Two edges are <u>incident</u> if there is exactly one vertex, shared by them.

Adjacency Matrix:

Every graph $G = (V, E)$ with $|V| = n$ has associated with it a symmetric adjacency matrix, which is a

Binary $n \times n$ Matrix A

in which

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \text{, i.e. } (v_i, v_j) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(v_i, v_j) = (v_j, v_i)$ $a_{ij} = a_{ji}$

$$A^T = A$$

The matrix A is a real-valued symmetric matrix.

The number of vertices adjacent to a given vertex v is called the degree of the vertex and is denoted

$$d(v) = |\{u \mid (u, v) \in E\}|$$

$$\sum_{v \in V} d(v) = 2|E| = 2m$$

Average degree of the graph

$$\bar{d} = \frac{\sum_{v \in V} d(v)}{|V|} = \frac{2m}{n}$$

- A graph is <u>complete</u> if all of its vertices are adjacent to all others.
- The extent to which a graph is complete is indicated by its <u>density</u> or <u>sparsity</u>.

Density = Number of edges divided by the number of possible total

$$= \frac{|E|}{\binom{n}{2}} = \frac{2m}{n(n-1)} \quad \text{(In strict graphs)}$$

$$= \frac{\sum d(v)}{n(n-1)}$$

$$= \frac{\bar{d}}{n-1}$$

- If a social network has many "well-connected people" then the network is "<u>dense</u>".

$$\bar{d} = \text{density}\ (n-1)$$

⇒ By connecting to a "well-connected <u>group</u> of individuals," one can belong to a "<u>subnetwork</u>" with high connectivity.

{ CLIQUE
CLAN
CLUB.

◇ A subgraph of a graph G is a graph whose vertices and edges are contained in G.

$$G_1 \subseteq G \quad \text{iff} \quad V_1 \subseteq V, \; E_1 \subseteq E.$$

◇ A clique is a maximal complete subgraph.

Clique $G_1 \subseteq G$, $|V_1| = k$, $|E_1| = \binom{k}{2}$

$\bar{d} = k-1$, (local) density = 1.

Thus, it's a maximal subgraph of G that has a density = 1.

= No extra node can be added to a clique without density decreasing below 1.

Example: PRIMARY GROUP: Nuclear Family,
Tribe.
Kindergarten Friends.

How can we capture the notion of an interesting subgraph, which, while not a clique, is still very "cliquish."

Professional Society, Social Clubs, ...
Not every pair of vertices is adjacent.
Yet, one could construct "short walks" of vertices between every pair of vertices.

WALK. A sequence of adjacent vertices
$v_0, v_1, \dots, v_n$ $[(v_i, v_{i+1}) \in E]$
is called a walk.

A walk can also be described by a sequence of incident edges.

PATH A walk in which no vertex occurs more than once is called a (simple) path.

TRAIL A walk in which no edge occurs more than once is called a (simple) trail.

$$\text{PATH} \underset{\subseteq}{\Rightarrow} \text{TRAIL} \underset{\subseteq}{\Rightarrow} \text{WALK}$$

Every path is a trail and every trail is a walk.

A walk is closed if $v_0 = v_n$, i.e. its start and end vertices coincide.

CYCLE A cycle is defined as a closed path in which $n \geq 3$.

CONNECTED COMPONENT.

A connected component of a graph is defined as a maximal subgraph in which path exists from every node to every other.

STRONGLY CONNECTED COMPONENT.

A strongly connected component of a graph is defined as a maximal subgraph in which cycle exists connecting every node to every other.

TREE : A tree is connected graph that contains no cycle.

→ Family Tree, Phylogeny Tree.

- In a tree every pair of vertices is connected by a unique path.
- Connected, but its strongly connected components are trivial singleton vertices.

◊ The <u>length of a walk</u> (resp. a trail or a path) is defined as the number of edges it contains.

◊ A walk between two vertices whose length is as short as possible is called a <u>geodesic</u> (or shortest path).

◊ The (geodesic)-<u>distance</u> between two vertices is the length of the shortest path connecting them. $\boxed{d(u,v) = \text{Geodesic Distance Between } u \,\&\, v.}$

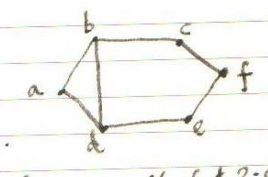
◊ The maximum geodesic-distance in a graph is its <u>DIAMETER</u>.

COHESIVE SUBSETS.

◊ An n-clique S of a graph is a maximal set of nodes in which

$$\forall_{u,v \in S} \quad d(u,v) \leq n$$

1-clique ≡ Clique



$\{a,b,d\}$ = 1-clique

$\{a,b,c,d,e\}$ = 2-clique.

Note $d(c,e) = 2$ thru f, even if $f \notin$ 2-clique.

- An n-clan is an n-clique in which the diameter of the ~~graph G~~ subgraph G(S) induced by S is less than or equal to n.

  (Note: The subgraph G(S) of a graph G induced by the set of nodes S is defined as the maximal subgraph of G that has vertex set S.)

An n-clan S is an n-clique in which all pairs of vertices are at distances less than or equal to n, even when the geodesics/paths are restricted to ~~and~~ involve only members of S.

{b, c, d, e, f} = 2-clan (& 2-clique)

{a, b, c, d, e} = 2-clique (but ≠ 2-clan)

n-clique ⊋ n-clan.

- An n-club is a subset S of nodes such that in the subgraph G(S) induced by S, the diameter is n or less.

{a, b, c, d, e} = 2-clique (but ≠ 2-club)

{a, b, c, d} = 2-club (but ≠ 2-clique not maximal)

COHESIVE SUBSETS.

n-clique → n-clan ← n-club

LS-Sets.

Let H be a set of nodes in $G=(V,E)$ and let K be a proper subset of H.

Let $\alpha(K)$ denote the number of edges linking members of K to $V \setminus K$.

Then H is an LS set of G if

$$\forall_{K \subsetneq H} \quad \alpha(K) > \alpha(H).$$

Members of H have more ties with the "insiders" (other members of H) than "outsiders" (members of $V \setminus H$).

Class Project.

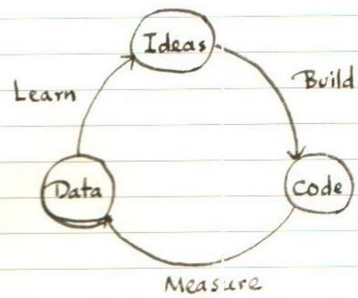
"Start up" Model.

→ LEAN STRUCTURE.

Create Value : Change the World

Create Social Goods

Disrupt.



5-WHYS

For want of a nail the shoe is lost.

shoe — horse
horse - rider
rider - battle
battle - kingdom

Team.

CEO. Overall Vision/Executive Summary/ Business Plan/Elevator Pitch.

(Investor →) Social Good

CSO. Science & Ideas / Intellectual Property/ Protection / Prior Art

(Patent Valuation)

CTO. Minimal Viable Product / Prototype/ Pivot... (Product Valuation)

COO. Collect Data / Analyze

(Clients Valuation).