Lecture #12

April 30 2013

ELECTION
(Crowd Sourcing)

- Collective Decision
  - Bargaining
  - Voting

◇ Aggregate individual preferences to estimate collective preference.

◇ How to combine dispersed information in a group.

Abstract Economy

1) $\mathcal{H}$ = A finite set of individuals.
   $H = |\mathcal{H}|$ = Number of individuals.

2) $\mathcal{P}$ = Policies [Candidates, choices] available to the group.

3) $i \in \mathcal{H}$, $p \in \mathcal{P}$ → Indirect utility function

$$u_i(p) = u(p, \alpha_i)$$

$\alpha_i \equiv$ Indexes the utility function.

BLISS POINT.

$$p^*(\alpha_i) = \arg\max_{p \in \mathcal{P}} u(p, \alpha_i)$$

How can one obtain a p^{**} from $\{p^*(\alpha_i) \mid i \in \mathcal{H}\}$?

Obvious Properties: a) p^{**} should not change if some irrelevant p is added to $\mathcal{P}$.

b) $\forall_{i \in \mathcal{H}}\ p^*(\alpha_i) = q \Rightarrow p^{**} = q$

c) Transitivity + Weakly Paretian, etc.

PREFERENCES (General Structure)

Complete, Reflexive and Transitive Preference Binary Relation

$\forall_{i \in \mathcal{H}}\ \forall_{z, z'} \quad z \gtrsim_i z' \ \vee\ z' \gtrsim_i z$ (Complete)

$\forall_{i \in \mathcal{H}}\ \forall_{z} \quad z \gtrsim_i z$ (Reflexive)

$\forall_{i \in \mathcal{H}}\ \forall_{z, z', z''} \quad z \gtrsim_i z' \wedge z' \gtrsim_i z'' \Rightarrow z \gtrsim_i z''$ (Transitive)

Aggregated (Welfare/Societal) Preference Relation

$\succsim_s$

(or $u^s(p)$)

satisfying certain properties or "AXIOMS."

AXIOM.

(A) IIA (Robust) Independent from irrelaivant alternative. $P \to P' = P \cup \{p_{irr}\}$ does not change the solution.

(B) PARETIAN (Unanimity) Unanimous preference agrees with social preference.
$\forall_{i \in H} \succsim_i$ :: same $\Rightarrow \succsim_i \equiv \succsim_s$

(C) SOVEREIGNITY (Non-imposition) Every possible social order should be achievable. (Transitivity)

Weakly Paretian + IIA
$\Rightarrow$ Intransitivity

{ Or Dictatorial
→ Set of Individuals with identical preferences...

CONDORCET PARADOX.

Or Problems with Majority Voting.

$\mathcal{H} = \{1, 2, 3\}$ ← 3 individual voters
$\mathcal{P} = \{a, b, c\}$ ← 3 candidate policies. (preferences)

① → $a > c > b$
② → $b > a > c$
③ → $c > b > a$

} Open Agenda Direct Democracy.

a, b and c get equal amount of votes → Majority?

Condorcet Paradox

	Restricted to b & c	Restricted to a & b
①	$c > b$	$a > b$
②	$b > c$	$b > a$
③	$c > b$	$b > a$

Majority Rule ⇒ $c \succsim_S b$ $\quad b \succsim_S a$

$\Rightarrow c \succsim_S a$ (By transitivity)

But both ① and ② prefere a to c and only 3 prefers $c > a$..

STRUCTURE

(A1) SINCERE VOTING:
Citizens vote truthfully and not strategically.

(A2) OPEN AGENDA:
Citizens vote over pairs of policy alternat[ives] in each round.
Winning strategy (preference) competes again[st] a new alternative in the next round.
→ Continue until no alternative is left.

(A3) DIRECT DEMOCRACY:
Citizens make policy choices directly through majority voting.

CONDORCET THEOREM

Open agenda direct democracy may not be able to select an aggregated preference relation.

ARROW'S IMPOSSIBILITY THEOREM. pp86
(More general than Condorcet & Sen)

◇ Set of feasible policies.

$$\mathcal{P} \subset \mathbb{R}^k$$

$\mathcal{R}$ = Set of all weak orders on $\mathcal{P}$.
← It contains information of the form (transitivity)

$$R_i \equiv p_1 \succsim_i p_2 \succsim_i \cdots \succsim_i p_k \succsim_i \cdots$$

All such individual orderings $R_i \in \mathcal{R}$.

SOCIETY (SOCIAL NETWORK):
$\mathcal{H}$ = Individuals

$$\rho = (R_1, R_2, \cdots, R_H) \in \mathcal{R}^H$$
= Preference Profile.

Restrictions. to $\mathcal{P}' \subset \mathcal{P}$.

$$\rho|_{\mathcal{P}'} = (R_1|_{\mathcal{P}'}, R_2|_{\mathcal{P}'}, \cdots, R_H|_{\mathcal{P}'})$$

= Preference profile where alternatives are restricted to $\mathcal{P}' \subset \mathcal{P}$.

(A) SOVEREIGNITY: (Unrestricted Domain)

$\mathcal{F} \subseteq \mathcal{P} \times \mathcal{P}$: Set of all Reflexive & Complete Binary Relationship on $\mathcal{P}$.

$\phi : \mathcal{R}^H \to \mathcal{F}$

Selects a Reflexive and Complete (but not necessarily Transitive) Binary Relationship on Policies → from Society's Preference Profile.

(B) UNANIMITY: (Weakly Paretian)

$$\left[\forall_{i \in H} \quad p \succsim_i p'\right] \Rightarrow \phi_H(p, p') \equiv p \succsim_H p'$$

(C) ROBUSTNESS: (Independent of Irrelevant Alternatives).

$$\forall_{\rho, \rho' \in \mathcal{R}^H} \quad \forall_{p, p' \in \mathcal{P}}$$

$$\rho\big|_{\{p, p'\}} = \rho'\big|_{\{p, p'\}}$$

$$\Rightarrow \phi(\rho)\big|_{\{p, p'\}} \equiv \phi(\rho')\big|_{\{p, p'\}}$$

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$\boxed{\text{IIA Axiom}}$ states that

If two preference profiles have same choice over two policy alternatives, then the social orderings that derive from these two preference profiles must also have identical choices over these two policy alternatives

~ Regardless of how these two preference profiles differ for other "Irrelevant Alternatives".

DECISIVE

Given $f \in \mathcal{R}^H$, a subset $\mathcal{D} \subseteq \mathcal{H}$ is decisive between $p, p' \in \mathcal{P}$, if

$$[\forall_{i \in \mathcal{D}}\ p \succsim_i p' \ \wedge \ \exists_{i \in \mathcal{D}}\ p \succ p'] \Rightarrow p \succsim_{\mathcal{H}} p'.$$

DICTATORIAL

$\mathcal{D} \subset \mathcal{H}$ is DICTATORIAL if

$\forall_{p, p' \in \mathcal{P}}$ $\mathcal{D}$ is decisive between p & p'

If $|\mathcal{D}| = 1$ (it is a singleton & dictatorial) then d, $(\mathcal{D} = \{d\})$ is a dictator. $\square$

ARROW'S IMPOSSIBILITY THEOREM.

Suppose that there are at least three alternati[ves]
Then if a social ordering, ϕ, is
Weakly Paretian,
Independent of Irrelevant Alternatives,
& Transitive
then it must be DICTATORIAL. $\square$

PROOF (TECHNICAL)
Sketch:

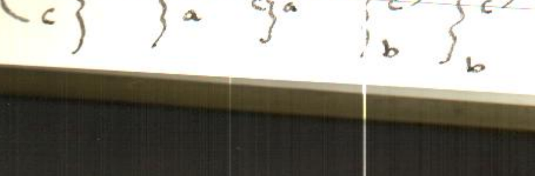
Think of three alternatives: a, b, c.
Choose b arbitrarily
Assume that in any profile you can only put b at the extreme (top or bottom).

$\Rightarrow$ Society must put b at the extreme (top or bottom)

Suppose not. Then society's choice is

$$a \underset{H}{\succsim} b \underset{H}{\succsim} c \quad \Rightarrow \quad a \underset{H}{\succsim} c$$

But we can then cleverly ~~[illegible]~~ reorganize the preferences such that a,b ordering and b,c orderings remain unchanged.



$\Rightarrow$ But now $\forall_i \; c \succ_i a$
$\Rightarrow c \underset{H}{\succ} a$
$\Rightarrow$ #

One can then construct a

PIVOTAL VOTER

who has the power to decide if b goes to the top or bottom. (While others only determine that b must ~~at~~ be at the extreme.)

$$h^* = H(b).$$

The set $\{h^*\} \subseteq H$ is a DICTATOR. ∎

TRUTHFUL VOTING

(Strategy Proof) $\left\{\right.$ Every individual has a dominant strategy (representing their preference truthfully.)

Strategy Proof $\Rightarrow \left\{ \begin{array}{l} \text{Weakly Paretian} \\ \text{IIA} \end{array} \right\} \Rightarrow \neg$Transitive or Dictatorial.

GIBBARD - SATTERWHITE THEOREM: