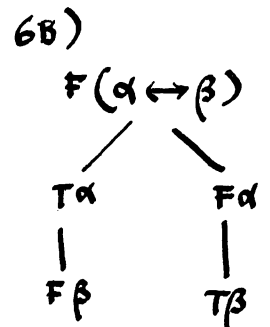
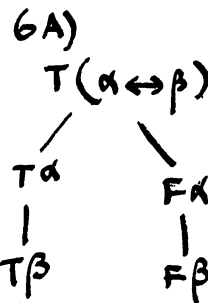
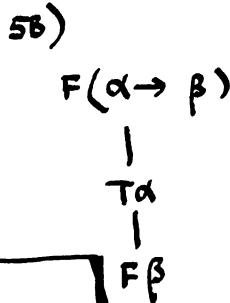
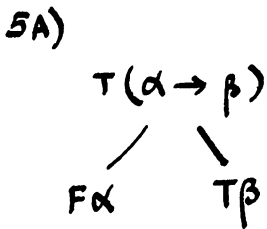
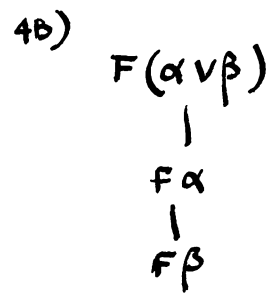
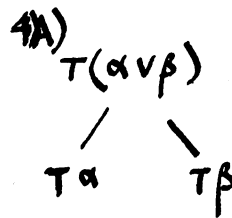
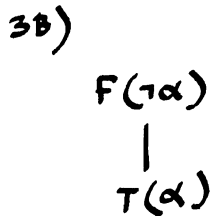
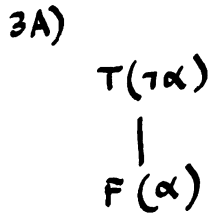
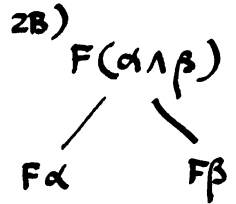
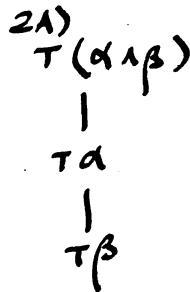
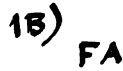
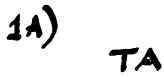


Oct 22, 2013

**LECTURE # 7**

(48)

Tableau Proof For Propositional Logic.



Thm (Pierce's Law)  
 $((A \rightarrow B) \rightarrow A) \rightarrow A$

Proof (By Tableau Method).

$F((A \rightarrow B) \rightarrow A)$

$T((A \rightarrow B) \rightarrow A)$

$F A$

$F(A \rightarrow B)$

$T A$

$T A$

$\perp$

$F B$

Thus.

$\neg((A \rightarrow B) \rightarrow A) \rightarrow A \Rightarrow \perp$

$\hookrightarrow \neg \text{SAT}$

$\therefore$  Its negation is valid.  $\square$

Tableau Method generalizes to first order logic.

# FIRST ORDER LOGIC.

49

→ More powerful than propositional logic.  
→ Expressive  
Computationally complex.

## Syntax:

Expressions in first-order logic are made up of a sequence of symbols.

- 1) Logical Symbols
- 2) Parameters (nonlogical symbols).

### Logical Symbols.

- ◊ Parantheses: (, )
- ◊ Propositional Symbols: (Connectives):  
 $\neg, \wedge$
- ◊ Variables:  $x_1, x_2, \dots$  (Countably many)
- ◊ Quantifiers:  $\forall$  (universal quantifier ForAll).

$\exists \equiv \neg \forall \neg$  : Existential quantifier  
There Exists

### Parameters

- ◊ Equality:  $=$   $x=y$
- ◊ Predicates:  $P(x)$   $x > y$
- ◊ Functions:  $f(x)$   $x+y$
- ◊ Constants:  $0, \pi, \text{bud.}$

↓  
Arity: #arguments used by predicates & functions.

$=$ ; has arity 2,  
2-ary predicate.

$\pi$ ; has arity 0,  
0-ary function.

## A FIRST-ORDER LANGUAGE

must specify its parameters (and is determined by it.)

# Examples..

(50)

	Prop. Logic.	Set Theory	Elementary Number Theory.
Equality.	No	Yes	Yes.
Predicates	$P_1, P_2, \dots$ (0-ary)	$\in$ (set membership)	$<$
Functions	No	None	$S, +, \times, \exp$
Constants.	No	$\emptyset$ (empty set)	$0$

~~Terms~~. TERMS, FORMULAS, WFF'S, BINDING (SCOPE). ...

Examples: First Order Theory of Numbers:

Model  $(\mathcal{N}, 0, S, +, \times)$

◊ Ordering Relation  $\{ \langle m, n \rangle \mid m < n \}$   
is defined by a wff:

$$v_1 < v_2 \text{ iff } \exists v_3 (v_1 + S v_3 = v_2)$$

$\downarrow$  Free variables.                       $\downarrow$  Bound variable

◊ Definability of a Natural Number:  $0, S0, SS0, \dots$

$$0 \in \mathcal{N}$$

$$v_1 \in \mathcal{N} \text{ iff } v_1 = 0 \text{ or } \exists v_2 \in \mathcal{N} \ v_1 = S v_2$$

◊ Primality:  $\mathcal{P} \subseteq \mathcal{N} \rightarrow$  Set of prime numbers:

$$v_1 \in \mathcal{P} \text{ iff } v_1 > 1 \wedge \forall v_2, v_3 (v_1 = v_2 \times v_3 \rightarrow v_2 = 1 \vee v_3 = 1)$$

◊ Note there exist relations on  $\mathcal{N}$  that are not definable.

TERMS:

For each function symbol  $f$  of arity  $n$  define a term building operation  $F_f$

$$F_f(\alpha_1, \alpha_2, \dots, \alpha_n) = f \alpha_1 \alpha_2 \dots \alpha_n$$

The set of terms is the set of expressions generated from the constant symbols and variables by the  $F_f$  operations:

$$\langle \alpha \rangle := \text{const} \mid v_1 \mid v_2 \mid \dots \mid F_f(\langle \alpha_1 \rangle, \langle \alpha_2 \rangle, \dots, \langle \alpha_n \rangle)$$

- ◊  $x_1 + x_2 \times x_3 \times x_3 + 4 \times x_4 \times x_4 \times x_4 \times x_4$
- ◊  $2 + x_1(4 + x_2(6 + x_3(8 + x_4)))$

FORMULAS:

Atomic Formula.

An atomic formula is an expression of the form:

$$P t_1 t_2 \dots t_n \equiv P t_1 t_2 \dots t_n$$

where  $P$  is a predicate symbol of arity  $n$  and  $t_1, t_2, \dots, t_n$  are terms.

~~Complex formulas~~

Formulas:

Defined by formula building operations:

$$E_1(\alpha) := \neg \alpha$$

$$E_2(\alpha, \beta) := \alpha \wedge \beta$$

$$Q_i(\alpha) := \forall v_i \alpha.$$

WFF (Well-Formed Formulas): The set of expressions generated from atomic formulas by the operations:

$$E_1, E_2, Q_i \quad (i=1, 2, \dots, n).$$

Free and Bound Variables:

A variable  $x$  occurs free in a wff  $\alpha$  iff

- ◊  $\alpha$  = atomic formula and occurs in  $\alpha$ .
- ◊  $\alpha \equiv \neg\beta$  and  $x$  occurs free in  $\beta$ .
- ◊  $\alpha \equiv \beta \wedge \gamma$  and  $x$  occurs free in  $\beta$  or in  $\gamma$ .
- ◊  $\alpha \equiv \forall v_i \beta$  and  $x$  occurs free in  $\beta$  and  $x \neq v_i$ .

(If  $\alpha \equiv \forall v_i \beta$  then  $v_i$  is said to be BOUND in  $\alpha$ ).

SENTENCE:

If no variable occurs free in wff  $\alpha$ , then  $\alpha$  is a sentence.

$\neg A)$   
 $\top \forall x \phi(x)$   
 $\quad \quad \quad \downarrow$   
 $\top \phi(a)$   
 (a is any parameter)

$\neg B)$   
 $\top \forall x \phi(x)$   
 $\quad \quad \quad \downarrow$   
 $\top \phi(a)$   
 (a is a new parameter)

$\delta A)$   
 $\top \exists x \phi(x)$   
 $\quad \quad \quad \downarrow$   
 $\top \phi(a)$   
 (a is a new parameter)

$\delta B)$   
 $\top \exists x \phi(x)$   
 $\quad \quad \quad \downarrow$   
 $\top \phi(a)$   
 (a is any parameter)

A parameter that has not been used before.

# Drinking Formula.

(53)

There is someone at a bar, when he drinks every one drinks.

$$F \exists x (Dx \rightarrow \forall y Dy) \quad [1]$$

$$F Da \rightarrow \forall y Dy \quad (\text{any } a)$$

$$T Da$$

$$F \forall y Dy$$

$$F Db$$

(new b)

$$F Db \rightarrow \forall y Dy \quad (\because [1])$$

$$T Db$$

$\perp$

Atomic Tableau: Tableau for a prime variable:  $\pi \in PV$ .

A finite Tableau: A binary tree labeled with signed propositions called entry satisfying the following inductive defn:

- (i) All atomic tableaux are finite tableaux.
- (ii) If  $\mathcal{T}$  is a finite tableau,  $P$  is a path on  $\mathcal{T}$ ,  $E$  is an entry of  $\mathcal{T}$  occurring on  $P$  and  $\mathcal{T}'$  is obtained from  $\mathcal{T}$  by adjoining the unique atomic tableau with root entry  $E$  to  $\mathcal{T}$  at the end of the path  $P$ , then  $\mathcal{T}'$  is also a finite tableau.

If  $\tau_0, \tau_1, \dots, \tau_n, \dots$  is a finite (or infinite) sequence of finite tableaux such that

$$\forall n \geq 0 \quad \tau_{n+1} \text{ is obtained from } \tau_n \text{ (by appen}^n \text{ of (ii))}$$

then  $\mathcal{T} = \bigcup \tau_n$  is a tableau.

$\mathcal{T}$  = Tableau,  
 $P$  = Path on  $\mathcal{T}$   
 $E$  = Entry on  $P$

$E$  = Reduced on  $P$ , if all entries on one path through the atomic tableau with root  $E$  occur on  $P$ .

$P$  = Contradicting if for some  $\text{prop}^n \alpha$ , both  $T\alpha$  and  $F\alpha$  entries occur on the path  $P$ .

$P$  = Finished  $\equiv$  Contradicting or all entries are reduced

$\mathcal{T}$  = Finished  $\forall P \in \mathcal{T} \quad P = \text{finished}$ .

$\mathcal{T}$  = Contradicting  $\forall P \in \mathcal{T} \quad P = \text{contradicting}$ .

A tableau proof of a wff  $\alpha$  is  
 $\equiv$  Contradictory tableau with root entry =  $F\alpha$ .  
 $\equiv (\neg\alpha \rightarrow \perp \equiv (\alpha \vee \perp) \equiv \alpha)$   
 $\equiv \alpha = \text{Valid.}$

SUBSTITUTION  $( )_t^x$  of some term  $t$  for a single variable  $x$ .

Formula resulting from replacing all free occurrences of  $x$  in  $\varphi$  by the term  $t$ .

$$\varphi^x_t \equiv \text{'}\varphi \text{ } t \text{ for } x\text{'}$$

$$\begin{aligned} x^x_t &\equiv t \\ y^x_t &\equiv y \quad (x \neq y). \\ c^x_t &\equiv c \quad c = \text{const.} \end{aligned}$$

$$\begin{aligned} (f \alpha_1 \alpha_2 \dots \alpha_n)^x_t &\equiv f \alpha_1^x_t \alpha_2^x_t \dots \alpha_n^x_t \\ (P t_1 t_2 \dots t_n)^x_t &\equiv P (t_1^x_t t_2^x_t \dots t_n^x_t) \end{aligned}$$

$$(t_1 = t_2)^x_t \equiv t_1^x_t = t_2^x_t$$

$$(\neg\alpha)^x_t \equiv \neg \alpha^x_t$$

$$(\alpha \wedge \beta)^x_t \equiv \alpha^x_t \wedge \beta^x_t$$

$$(\forall v_i \alpha)^x_t \equiv \begin{cases} \forall v_i \alpha \\ \forall v_i \alpha^x_t \end{cases}$$

### SIMULTANEOUS SUBST<sup>N</sup>

$$\varphi^{\begin{matrix} x_1 & x_2 & \dots & x_n \\ t_1 & t_2 & \dots & t_n \end{matrix}} \quad (x_1, x_2, \dots, x_n \text{ dist.})$$

The v'bles  $x_i$  are simultaneously replaced by the terms  $t_i$  at free occurrences.

$x \equiv$  Bound vbl  
 $= v_i$

$x =$  free.