Oct 22, 2013

LECTURE #7

Tableu Proof For Propositional Logic.

1A) TA

1B) FA

2A) $T(\alpha \wedge \beta)$
- $T\alpha$
  - $T\beta$

2B) $F(\alpha \wedge \beta)$
- $F\alpha$
- $F\beta$

3A) $T(\neg\alpha)$
- $F(\alpha)$

3B) $F(\neg\alpha)$
- $T(\alpha)$

4A) $T(\alpha \vee \beta)$
- $T\alpha$
- $T\beta$

4B) $F(\alpha \vee \beta)$
- $F\alpha$
  - $F\beta$

5A) $T(\alpha \to \beta)$
- $F\alpha$
- $T\beta$

5B) $F(\alpha \to \beta)$
- $T\alpha$
  - $F\beta$

6A) $T(\alpha \leftrightarrow \beta)$
- $T\alpha$
  - $T\beta$
- $F\alpha$
  - $F\beta$

6B) $F(\alpha \leftrightarrow \beta)$
- $T\alpha$
  - $F\beta$
- $F\alpha$
  - $T\beta$

<u>Thm</u> (Pierce's Law)

$$(((A \to B) \to A) \to A)$$

Proof (By Tableu Method).

- $F(((A \to B) \to A) \to A)$
  - $T((A \to B) \to A)$
    - FA
      - $F(A \to B)$
        - TA
          - FB
      - TA
        - $\perp$

Thus.

$$\neg(((A \to B) \to A) \to A) \Rightarrow \perp$$

$\hookrightarrow \neg SAT$

$\therefore$ Its negation is valid. $\blacksquare$

Tableu Method generalizes to first order logic.

FIRST ORDER LOGIC.

→ More powerful than propositional logic.

→ Expressive
computationally complex.

Syntax:

Expressions in first-order logic are made up of a sequence of symbols.

1) Logical Symbols
2) Parameters (nonlogical symbols).

Logical Symbols.

- Parantheses: (,)
- Propositonal Symbols: (Connectives): $\neg$, $\wedge$
- Variables: $v_1, v_2, \cdots$ (Countably many).
- Quantifiers: $\forall$ (universal quantifier ForAll).

$\exists \equiv \neg\forall\neg$: Existential quantifier
There Exists

Parameters

- Equality: $=$ $\quad x=y$
- Predicates: $p(x)$ $\quad x>y$
- Functions: $f(x)$ $\quad x+y$
- Constants: $0, \pi$, bud.

⇓

Arity: #arguments used by predicates & functions.

$=$; has arity 2,
2-ary predicate.

π; has arity 0,
0-ary function.

A FIRST-ORDER LANGUAGE
must specify its parameters (and is determined by it.)

Examples..

	Prop. Logic.	Set Theory	Elementary Number Theory.
Equality.	No	Yes	Yes.
Predicates	$p_1, p_2, \cdots$ (0-ary)	$\in$ (set membership)	$<$
Functions	No	None	$S, +, \times, \exp$
Constants.	No	ϕ (empty set)	0

~~Terms~~. TERMS, FORMULAS, WFF's, BINDING (SCOPE). ...

Examples: First Order Theory of Numbers:

Model $(\mathcal{N}, 0, S, +, \times)$

◇ Ordering Relation $\{\langle m,n\rangle \mid m<n\}$ is defined by a wff:

$$v_1 < v_2 \quad \text{iff} \quad \exists v_3 \; (v_1 + Sv_3 = v_2)$$

Free variables. Bound variable

◇ Definability of a Natural Number: $0, S0, SS0, \ldots$

$0 \in \mathcal{N}$

$v_1 \in \mathcal{N}$ iff $v_1 = 0$ or $\exists v_2 \in \mathcal{N} \;\; v_1 = Sv_2$

◇ Primality: $\mathcal{P} \subseteq \mathcal{N}$ → Set of prime numbers:

$$v_1 \in \mathcal{P} \quad \text{iff} \quad v_1 > 1 \;\wedge\; \forall v_2, v_3 \; (v_1 = v_2 \times v_3 \rightarrow v_2 = 1 \vee v_3 = 1)$$

◇ Note there exist relations on $\mathcal{N}$ that are not definable.

TERMS:

For each function symbol f of arity n define a term building operation $\mathcal{F}_f$

$$\mathcal{F}_f(\alpha_1, \alpha_2, \ldots, \alpha_n) = f\alpha_1\,\alpha_2 \cdots \alpha_n$$

The set of terms is the set of expressions generated from the constant symbols and variables by the $\mathcal{F}_f$ operations:

$$\langle\alpha\rangle \;::=\; \text{const} \mid v_1 \mid v_2 \mid \cdots \mid \mathcal{F}_f(\langle\alpha_1\rangle, \langle\alpha_2\rangle, \ldots \langle\alpha_n\rangle)$$

- $x_1 + x_2 \times x_3 \times x_3 + 4 \times x_4 \times x_4 \times x_4 \times x_4$
- $2 + x \times (4 + x \times (6 + x \times (8 + x)))$

FORMULAS:

Atomic Formula.

An atomic formula is an expression of the form:

$$Pt_1\,t_2 \cdots t_n \equiv Pt_1\,t_2 \cdots t_n$$

where P is a predicate symbol of arity n and $t_1, t_2, \ldots t_n$ are terms.

Formulas:

Defined by formula building operations:

$$\mathcal{E}_\neg(\alpha) := \neg\alpha$$
$$\mathcal{E}_\wedge(\alpha, \beta) := \alpha \wedge \beta$$
$$Q_i(\alpha) := \forall v_i\, \alpha.$$

WFF (Well-Formed Formulas): The set of expressions generated from atomic formulas by the operations:

$$\mathcal{E}_\neg, \mathcal{E}_\wedge, Q_i \quad (i = 1, 2, \ldots n).$$

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Free and Bound Variables:

A variable x occurs *free* in a wff α iff

- α = atomic formula and occurs in α.
- $\alpha \equiv \neg\beta$ and x occurs free in β.
- $\alpha \equiv \beta \wedge \gamma$ and x occurs free in β or in γ.
- $\alpha \equiv \forall v_i \beta$ and x occurs free in β and $x \not\equiv v_i$.

(If $\alpha \equiv \forall v_i \beta$ then v_i is said to be **BOUND** in α).

SENTENCE:

If no variable occurs free in wff α, then α is a *sentence*.

7A)	7B)	8A)	8B)
$T \forall x\, \phi(x)$	$F \forall x\, \phi(x)$	$T \exists x\, \phi(x)$	$F \exists x\, \phi(x)$
$\vert$	$\vert$	$\vert$	$\vert$
$T \phi(a)$	$F \phi(a)$	$T \phi(a)$	$F \phi(a)$
(a is any parameter)	(a is a *new* parameter)	(a is a *new* parameter)	(a is any parameter)

A parameter that has not been used before.

Drinking Formula.

There is someone at a bar, when he drinks every one drinks.

$$F\ \exists x\,(Dx \to \forall y\, Dy) \qquad [1]$$

$$F\ Da \to \forall y\, Dy \qquad (\text{any } a)$$

$$T\ Da$$

$$F\ \forall y\, Dy$$

$$F\ Db \qquad (\text{new } b)$$

$$F\ Db \to \forall y\, Dy \qquad (\because [1])$$

$$T\ Db$$

$$\perp$$

$$\sim$$

<u>Atomic Tableau</u>: Tableau for a prime variable : $\pi \in PV$.

<u>A finite Tableau</u>: A binary tree labeled with signed propositions called <u>entry</u> satisfying the following inductive defn:

(i) All atomic tableau are finite tableau.
(ii) If τ is a finite tableau, P is a path on τ, E is an entry of τ occurring on P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also a finite tableau.

If $\tau_0, \tau_1, \dots \tau_n, \dots$ is a finite (or infinite) sequence of finite tableau such that

$\forall n \geq 0$ τ_{n+1} is obtained from τ_n (by appl^n of (ii))

then $\tau = \bigcup \tau_n$ is a tableau.

τ = Tableau,
P = Path on τ
E = Entry on P

E = <u>Reduced</u> on P, if all entries on one path through the atomic tableau with root E occur on P.

P : <u>Contradictory</u> if for some ~~poof~~^{n} α, both $T\alpha$ and $F\alpha$ entries occur on the path P.

P = <u>Finished</u> $\equiv$ Contradictory or all entries are reduced

τ : <u>Finished</u> $\forall_{P \in \tau}$ P = finished.

= <u>Contradictory</u> $\forall_{P \in \tau}$ P : contradictory

A tableu proof of a wff α is

$\equiv$ Contradictory tableu with root entry $= F\alpha$.

$\equiv \left(\neg\alpha \to \bot \;\; \equiv \;\; (\alpha \vee \bot) \equiv \alpha \right)$

$\equiv \alpha$: Valid.

<u>SUBSTITUTION</u> $(\;)^x_t$ of some term t for a single variable x.

Formula resulting from replacing all free occurrences of x in φ by the term t.

$$\varphi^x_t \equiv \text{'}\varphi \; t \text{ for } x\text{'}$$

$x^x_t \equiv t$

$y^x_t \equiv y \quad (x \neq y)$.

$c^x_t \equiv c \qquad c = \text{const}$.

$(f\,\alpha_1\,\alpha_2 \ldots \alpha_n)^x_t$
$\equiv f\,\alpha_1{}^x_t\,\alpha_2{}^x_t \ldots \alpha_n{}^x_t$

$(P\,t_1\,t_2 \ldots t_n)^x_t$ -
$\equiv P\,(t_1{}^x_t\,t_2{}^x_t \ldots t_n{}^x_t)$

$(t_1 = t_2)^x_t \equiv t_1{}^x_t = t_2{}^x_t$

$(\neg\alpha)^x_t \equiv \neg\,\alpha^x_t$

$(\alpha \wedge \beta)^x_t \equiv \alpha^x_t \wedge \beta^x_t$

SIMULTANEOUS SUBST$^{N.}$

$\varphi^{x_1\,x_2\,\ldots\,x_n}_{t_1\,t_2\,\ldots\,t_n}$

($x_1, x_2, \ldots x_n$ dist.)

The v'bles x_i are simultaneously replaced by the terms t_i at free occurrences.

$$(\forall v_i\,\alpha)^x_t \equiv \begin{cases} \forall v_i\,\alpha & x \equiv \text{Bound vble} = v_i \\ \forall v_i\,\alpha^x_t & x = \text{free}. \end{cases}$$