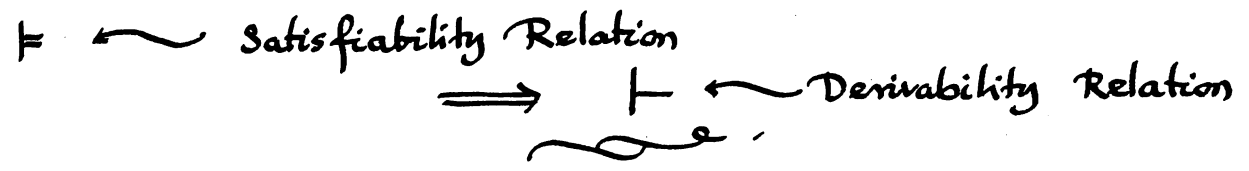


Oct 1 2013

LECTURE #5

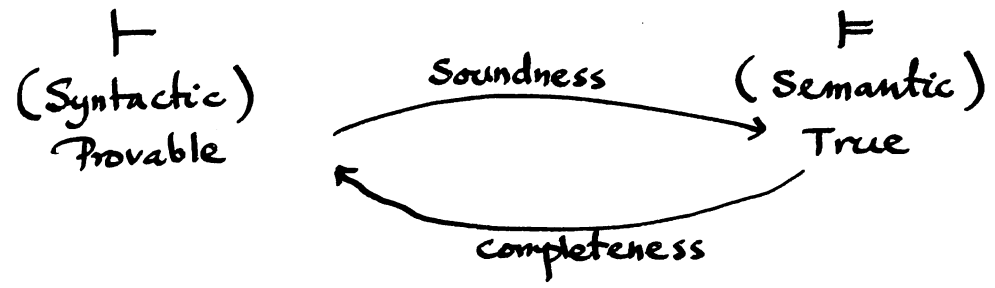
A Calculus of Natural Deduction:

Purely Syntactic Rules \rightarrow Formal Logic.



Gentzen Type Calculus.

Hilbert Type Calculus.



Pairs of Formulas $\left\{ \begin{matrix} X \\ \alpha \end{matrix} \right\}$, with respect to which we may fix some rules.

$X, \alpha \equiv$ sequenz/sequent

$X \vdash \alpha$ $\{ \alpha$ is derivable (provable) from X



BASIC RULES.

(IS)
Initial Sequent

$$\frac{}{\alpha \vdash \alpha}$$

(MR)
Monotonicity

$$\frac{X \vdash \alpha}{X' \vdash \alpha}, \quad X' \supseteq X$$

(11)
$$\frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$$

~~(11)~~ (11)
$$\frac{X \vdash \alpha, \neg \alpha}{X \vdash \beta}$$

(12)
$$\frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$$

(12)
$$\frac{X, \alpha \vdash \beta \quad X, \neg \alpha \vdash \beta}{X \vdash \beta}$$

Derivation \equiv A finite sequent $(S_0; S_1; \dots; S_n)$
 $S_n = (X, \alpha)$

Claim 1. $\alpha, \beta \vdash \alpha \wedge \beta$

$$\frac{\frac{\frac{}{\alpha \vdash \alpha}}{\alpha, \beta \vdash \alpha} \text{ MR} \quad \frac{\frac{}{\beta \vdash \beta}}{\alpha, \beta \vdash \beta} \text{ MR}}{\alpha, \beta \vdash \alpha \wedge \beta} \text{ 11}}$$

Claim 2. \neg elimination

$$\frac{X, \neg \alpha \vdash \alpha}{X \vdash \alpha}$$

$$\frac{X, \neg \alpha \vdash \alpha \quad \frac{\frac{}{\alpha \vdash \alpha}}{X, \alpha \vdash \alpha} \text{ MR}}{X \vdash \alpha} \text{ 12}$$

Claim 3. Reductio ad absurdum

$$\frac{X, \neg \alpha \vdash \beta, \neg \beta}{X \vdash \alpha}$$

$$\frac{\frac{X, \neg \alpha \vdash \beta, \neg \beta}{X, \neg \alpha \vdash \alpha} \text{ 11}}{X \vdash \alpha} \text{ 12 elim}$$

Claim 4 \rightarrow Elimination

$$\frac{X \vdash \alpha \rightarrow \beta}{X, \alpha \vdash \beta}$$

(37) (H)

$$\frac{\frac{X, \alpha, \neg\beta \vdash \alpha, \neg\beta \quad (11)}{X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta} \quad \frac{X \vdash \neg(\alpha \wedge \neg\beta)}{X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)} \quad (11)}{X, \alpha, \neg\beta \vdash \beta} \quad (\neg\text{elim})$$

$$\frac{}{X, \alpha \vdash \beta}$$

Claim 5 \rightarrow CUT

$$\frac{X \vdash \alpha \quad X, \alpha \vdash \beta}{X \vdash \beta}$$

(H) (MR) (MR) (11) (12)

$$\frac{\frac{X \vdash \alpha \quad (H)}{X, \neg\alpha \vdash \alpha} \quad (MR) \quad \frac{\frac{}{\neg\alpha \vdash \neg\alpha} \quad IS}{X, \neg\alpha \vdash \neg\alpha} \quad MR}{X, \neg\alpha \vdash \beta} \quad (11)$$

$$\frac{X, \neg\alpha \vdash \beta \quad (H) \quad X, \neg\alpha \vdash \neg\alpha}{X \vdash \beta} \quad (12)$$

Claim 6 \rightarrow DETACHMENT

$$\frac{X \vdash \alpha, \alpha \rightarrow \beta}{X \vdash \beta}$$

(H) (\rightarrow elim) CUT

$$\frac{X \vdash \alpha \quad (H) \quad \frac{X \vdash \alpha \rightarrow \beta}{X, \alpha \vdash \beta} \quad (\rightarrow \text{elim})}{X \vdash \beta} \quad \text{CUT}$$

COROLLARY: MP: Modus Ponens.

$\alpha, \alpha \rightarrow \beta \vdash \beta$

IS
DET

$$\frac{\frac{}{\alpha, \alpha \rightarrow \beta \vdash \alpha, \alpha \rightarrow \beta} \quad IS}{\alpha, \alpha \rightarrow \beta \vdash \beta} \quad DET$$

RULE (Rule-based System)

R : $\frac{X_1 \vdash \alpha_1 ; X_2 \vdash \alpha_2 ; \dots ; X_n \vdash \alpha_n}{X \vdash \alpha}$

Example:

DET: $\frac{X \vdash \alpha, \alpha \rightarrow \beta}{X \vdash \beta}$

Gentzen Style Rules.

PROPERTIES THAT ARE CLOSED UNDER \mathcal{R} .

(38)

A property \mathcal{E} which is closed under \mathcal{R}

$\mathcal{E}(X_1, \alpha_1); \mathcal{E}(X_2, \alpha_2); \dots \mathcal{E}(X_n, \alpha_n)$ implies $\mathcal{E}(X, \alpha)$

SOUNDNESS PROPERTY $\mathcal{E}(X, \alpha) \stackrel{\circ}{=} X \models \alpha$

It is closed under the basic rules of \vdash .

(5) Prove that $X \vdash \alpha$ implies $X \models \alpha$

That is $\forall \omega \ \omega \models X \rightarrow \omega \models \alpha$

All models of X are models α , if α is provable from X .

Proof-theoretic "truth" \subseteq Model-theoretic "truth".

$\mathcal{E}: X \models \alpha$, this property applies to all provable sequents.

Thus, the relation \vdash is semantically sound.

PRINCIPLE OF RULE INDUCTION:

Let $\mathcal{E} (\subseteq \mathcal{P}\mathcal{F} \rightarrow \mathcal{F})$ be a property closed under all basic rules of \vdash . Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

Proof: By induction on the length of a derivation $S = (X, \alpha)$.

SOUNDNESS:

" $\vdash \subseteq \models$ "

More explicitly,

$$x \vdash \alpha \Rightarrow x \models \alpha \quad \forall x, \alpha.$$

Proof: (Only show for $x = \text{finite}$.)

$$(IS) \quad \forall \omega \quad \omega \models \alpha \Rightarrow \omega \models \alpha.$$

$$\therefore \alpha \vdash \alpha \Rightarrow \alpha \models \alpha.$$

$$(MR) \quad x' \supseteq x$$

$$\forall \omega \quad \omega \models x' \Rightarrow \omega \models x$$

$$\forall \omega \quad (\omega \models x \Rightarrow \omega \models \alpha) \Rightarrow \forall \omega \quad (\omega \models x' \Rightarrow \omega \models \alpha)$$

$$\frac{x \vdash \alpha}{x' \vdash \alpha} (x' \supseteq x) \Rightarrow x \models \alpha \text{ implies } x' \models \alpha.$$

$$(11) \quad \forall \omega \quad (\omega \models x \Rightarrow \omega \models \alpha \text{ and } \omega \models \beta) \Rightarrow \forall \omega \quad (\omega \models x \Rightarrow \omega \models \alpha \wedge \beta)$$

$$\frac{x \vdash \alpha, \beta}{x \vdash \alpha \wedge \beta} \Rightarrow x \models \alpha, \beta \text{ implies } x \models \alpha \wedge \beta.$$

$$(12) \quad \forall \omega \quad (\omega \models x \Rightarrow \omega \models \alpha \wedge \beta) \Rightarrow \forall \omega \quad (\omega \models x \Rightarrow \omega \models \alpha, \omega \models \beta)$$

$$\frac{x \vdash \alpha \wedge \beta}{x \vdash \alpha, \beta} \Rightarrow x \models \alpha \wedge \beta \text{ implies } x \models \alpha, \beta.$$

$$(T1) \quad \forall \omega \quad (\omega \models x \Rightarrow \omega \models \alpha, \omega \models \neg \alpha) \Rightarrow \forall \omega \quad \omega \not\models x \quad (\Leftrightarrow \neg \exists \omega \omega \models x)$$

Vacuously $\forall \omega \quad \omega \models x \text{ implies } \omega \models \beta.$

$$\frac{x \vdash \alpha, \neg \alpha}{x \vdash \beta} \Rightarrow x \models \alpha, \neg \alpha \text{ implies } x \models \beta.$$

$$(T2) \quad \forall \omega \quad (\omega \models x, \alpha \text{ and } \omega \models x, \neg \alpha \Rightarrow \omega \models \beta)$$

$$\Rightarrow \forall \omega \quad \omega \models x \Rightarrow \omega \models \beta.$$

$$\frac{x, \alpha \vdash \beta \quad x, \neg \alpha \vdash \beta}{x \vdash \beta} \Rightarrow x, \alpha \models \beta \text{ and } x, \neg \alpha \models \beta \text{ implies } x \models \beta.$$

FINITENESS THEOREM: for \vdash

If $X \vdash \alpha$, then there is a finite subset $X_0 \subseteq X$ with $X_0 \vdash \alpha$. (40)

Proof: By induction

Let $E^F(x, \alpha)$ be the property.

$$\boxed{\exists X_0 \subseteq X, X_0 = \text{finite} \quad X_0 \vdash \alpha}$$

Show that E^F is closed under the basic rule. \square .

