Oct 1 2013

$\boxed{\text{LECTURE \#5}}$

A Calculus of Natural Deduction:

Purely Syntactic Rules $\longrightarrow$ Formal Logic.

$\models$ $\longleftarrow$ Satisfiability Relation

$\Longrightarrow$ $\vdash$ $\longleftarrow$ Derivability Relation

<u>Gentzen Type Calculus.</u> <u>Hilbert Type Calculus.</u>

$$\begin{tikzcd} \begin{array}{c} \vdash \\ \text{(Syntactic)} \\ \text{Provable} \end{array} \arrow[r, bend left, "\text{Soundness}"] & \begin{array}{c} \models \\ \text{(Semantic)} \\ \text{True} \end{array} \arrow[l, bend left, "\text{completeness}"] \end{tikzcd}$$

Pairs of Formulas $\begin{Bmatrix} X \\ \alpha \end{Bmatrix}$, with respect to which we may <u>fix</u> some rules.

X, α $\equiv$ Sequenz / Sequent

$X \vdash \alpha$ $\{$ α is derivable (provable) from X

BASIC RULES.

(IS) Initial Sequent $\quad \dfrac{}{\alpha \vdash \alpha}$

(MR) Monotonicity $\quad \dfrac{X \vdash \alpha}{X' \vdash \alpha}, \quad X' \supseteq X$

($\wedge 1$) $\quad \dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$

($\wedge 2$) $\quad \dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$

($\neg 1$) $\quad \dfrac{X \vdash \alpha, \neg\alpha}{X \vdash \beta}$

($\neg 2$) $\quad \dfrac{X, \alpha \vdash \beta \qquad X, \neg\alpha \vdash \beta}{X \vdash \beta}$

Derivation $\equiv$ A finite sequent $(S_0; S_1; \cdots; S_n)$
$S_n = (X, \alpha)$

Claim 1. $\alpha, \beta \vdash \alpha \wedge \beta$

$$\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,\text{IS}}{\alpha, \beta \vdash \alpha}\,\text{MR} \qquad \dfrac{\dfrac{}{\beta \vdash \beta}\,\text{IS}}{\alpha, \beta \vdash \beta}\,\text{MR}}{\alpha, \beta \vdash \alpha \wedge \beta}\,\wedge 1$$

Claim 2. $\neg$ elimination
$$\dfrac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$$

$$\dfrac{X, \neg\alpha \vdash \alpha \;(\#) \qquad \dfrac{\dfrac{}{\alpha \vdash \alpha}\,\text{IS}}{X, \alpha \vdash \alpha}\,\text{MR}}{X \vdash \alpha}\,\neg 2$$

Claim 3. Reductio ad absurdum
$$\dfrac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$$

$$\dfrac{\dfrac{X, \neg\alpha \vdash \beta, \neg\beta}{X, \neg\alpha \vdash \alpha}\,\neg 1}{X \vdash \alpha}\,\neg\,\text{elim}$$

Claim 4 → Elimination

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}$$

$$\dfrac{\dfrac{X, \alpha, \neg\beta \vdash \alpha, \neg\beta}{X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta}\,(\wedge 1) \qquad \dfrac{X \vdash \neg(\alpha \wedge \neg\beta)\ (H)}{X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)}}{\dfrac{X, \alpha, \neg\beta \vdash \beta}{X, \alpha \vdash \beta}\,(\neg\text{elim})}\,(\neg 1)$$

claim 5 → CUT

$$\frac{X \vdash \alpha \qquad X, \alpha \vdash \beta}{X \vdash \beta}$$

$$\dfrac{(H)\ X, \alpha \vdash \beta \qquad \dfrac{\dfrac{X \vdash \alpha\ (H)}{X, \neg\alpha \vdash \alpha}\,(MR) \qquad \dfrac{\overline{\neg\alpha \vdash \neg\alpha}\ IS}{X, \neg\alpha \vdash \neg\alpha}\,MR}{X, \neg\alpha \vdash \beta}\,(\neg 1)}{X \vdash \beta}\,(\neg 2)$$

Claim 6 → DETACHMENT

$$\frac{X \vdash \alpha,\ \alpha \to \beta}{X \vdash \beta}$$

$$\dfrac{X \vdash \alpha\ (H) \qquad \dfrac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}\,(\to\text{elim})}{X \vdash \beta}\,\text{CUT}$$

COROLLARY: MP: Modus Ponen.

$$\alpha, \alpha \to \beta \vdash \beta$$

$$\dfrac{\overline{\alpha, \alpha \to \beta \vdash \alpha, \alpha \to \beta}\ IS}{\alpha, \alpha \to \beta \vdash \beta}\,DET$$

RULE (Rule-based System)

$$R: \quad \frac{X_1 \vdash \alpha_1;\quad X_2 \vdash \alpha_2;\quad \ldots\quad ;\ X_n \vdash \alpha_n}{X \vdash \alpha.}$$

Example:

$$DET: \quad \frac{X \vdash \alpha,\ \alpha \to \beta}{X \vdash \beta}$$

Gentzen Style Rules.

PROPERTIES THAT ARE CLOSED UNDER $\mathcal{R}$.

A property $\mathcal{E}$ which is closed under $\mathcal{R}$

$$\mathcal{E}(X_1, \alpha_1);\ \mathcal{E}(X_2, \alpha_2);\ \cdots\ \mathcal{E}(X_n, \alpha_n) \text{ implies } \mathcal{E}(X, \alpha)$$

SOUNDNESS PROPERTY $\quad \mathcal{E}(X, \alpha) \stackrel{\circ}{=} X \models \alpha$

It is closed under the basic rules of $\vdash$.

(S) Prove that $X \vdash \alpha$ implies $X \models \alpha$

That is $\forall \omega \quad \omega \models X \rightarrow \omega \models \alpha$

All models of X are models α, if α is provable from X.

Proof-theoretic "truth" $\subseteq$ Model-theoretic "truth".

$\mathcal{E}: X \models \alpha$, this property applies to all provable sequents.

Thus, the relation $\vdash$ is semantically sound.

PRINCIPLE OF RULE INDUCTION:

Let $\mathcal{E}$ ($\subseteq \mathcal{PF} \rightarrow \mathcal{F}$) be a property closed under all basic
$: X \mapsto \alpha$
rules of $\vdash$. Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$.

Proof: By induction on the length of a derivation $S = (X, \alpha)$.
$\square$

SOUNDNESS:

$$\text{“} \vdash \subseteq \models \text{”}$$

More explicitly,

$$X \vdash \alpha \Rightarrow X \models \alpha \qquad \forall X, \alpha.$$

Proof: (Only show for X = finite.)

(IS) $\forall_\omega \; \omega \models \alpha \Rightarrow \omega \models \alpha.$

$\therefore \; \alpha \vdash \alpha \Rightarrow \alpha \models \alpha.$

(MR) $X' \supseteq X$

$\forall_\omega \; \omega \models X' \Rightarrow \omega \models X$

$\forall_\omega \; (\omega \models X \Rightarrow \omega \models \alpha) \Rightarrow \forall_\omega \; (\omega \models X' \Rightarrow \omega \models \alpha)$

$$\frac{X \vdash \alpha}{X' \vdash \alpha} \; (X' \supseteq X) \Rightarrow \quad X \models \alpha \text{ implies } X' \models \alpha.$$

(∧1) $\forall_\omega \; (\omega \models X \Rightarrow \omega \models \alpha \text{ and } \omega \models \beta) \Rightarrow \forall_\omega \; (\omega \models X \Rightarrow \omega \models \alpha \wedge \beta)$

$$\frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} \Rightarrow \quad X \models \alpha, \beta \text{ implies } X \models \alpha \wedge \beta.$$

(∧2) $\forall_\omega \; (\omega \models X \Rightarrow \omega \models \alpha \wedge \beta) \Rightarrow \forall_\omega \; (\omega \models X \Rightarrow \omega \models \alpha, \; \omega \models \beta)$

$$\frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta} \Rightarrow \quad X \models \alpha \wedge \beta \text{ implies } X \models \alpha, \beta.$$

(¬1) $\forall_\omega \; (\omega \models X \Rightarrow \omega \models \alpha, \; \omega \models \neg\alpha) \Rightarrow \forall_\omega \; \omega \not\models X \; (\Leftrightarrow \neg \exists_\omega \; \omega \models X)$

Vacuously $\forall_\omega \; \omega \models X$ implies $\omega \models \beta$.

$$\frac{X \vdash \alpha, \neg\alpha}{X \vdash \beta} \Rightarrow \quad X \models \alpha, \neg\alpha \text{ implies } X \models \beta.$$

(¬2) $\forall_\omega \; (\omega \models X, \alpha \text{ and } \omega \models X, \neg\alpha \Rightarrow \omega \models \beta)$

$\Rightarrow \forall_\omega \; \omega \models X \Rightarrow \omega \models \beta.$

$$\frac{X, \alpha \vdash \beta \quad X, \neg\alpha \vdash \beta}{X \vdash \beta} \Rightarrow \quad X, \alpha \models \beta \text{ and } X, \neg\alpha \models \beta \text{ implies } X \models \beta.$$

FINITENESS THEOREM: for $\vdash$

If $X \vdash \alpha$, then there is a finite subset $X_0 \subseteq X$ with

$$X_0 \vdash \alpha.$$

Proof: By induction

Let $\mathcal{E}^F(X,\alpha)$ be the property

$$\boxed{\exists X_0 \subseteq X,\ X_0 = \text{finite} \qquad X_0 \vdash \alpha}$$

Show that $\mathcal{E}^F$ is closed under the basic rule. □