N O T E S
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Game Theory

A social network is not a static structure – the individuals in a social network must constantly interact in order to create "social capital" that reflects how a group may be able to achieve much more than just the sum of what each individual achieves. The interaction involves making certain kinds of choices; see below:

- Share information
- Evaluate information (obtained from friends, acquaintances and coworkers)
- Develop trust
- Accept or reject friendship
- Recommend friendship
- Buy and sell goods from other individuals
- Bid in an auction
- Bargain
- Visit a website

How an individual may make his choices can be formulated within the classical "game theory:" with simple assumptions such as knowledge of one's choices (strategies), knowledge of one's pay-offs (utilities), individuals being rational and CKR (common-knowledge of rationality). Thus a method of studying strategic decision-making can be studied within the following possible frameowrks

- Static Games
- Dynamic Games – Repeated (unbounded) Games under Uncertainty
  - Bargaining Games
  - Signaling Games
- Evolutionary Games

There are few assumptions to be made: (1) The individuals in the game know their choices or *strategies*; (2) The individuals know their payoff functions or *utility*; and (3) The individuals involved in the interactions are not only *rational* – utility-maximizing – but also they know that they are rational, and that their opponents know that they are rational, and that they know that they are rational and that their opponents know that they are rational and so on. This kind of recursive reasoning will be called *common knowledge of rationality*: CKR.

Thus these individuals act rationally in the sense of choosing an option that gives them higher pay-offs (pleasure-seeking-pain-avoiding), and their actions reveal everything about them, in the sense that any other activities by them are deemed immaterial (e.g., verbal promises, etc.) Thus we may assume that they are non-cooperative in the sense that they do nothing other than seeking highest pay-offs selfishly as determined by their rationality, their knowledge of pay-offs and choices, etc.

- Payoffs need not be just monetary – social and psychological payoffs may matter to the individuals as would be revealed by their actions (e.g., occasionally altruistic).
- However, still rational-decision making paradigm remains useful in providing a foundation for the theory. These ideas can be further constrained by epistemological and cognitive limitations, as developed in *theories of bounded rationality*, where individual's choice need to be only good-enough or statisficing or *theories of evolutionary games*, where individual's choices determine their reproductive fitness, thus allowing only optimizing individuals to survive.

We will start with certain ordinal information when there is one individual optimizing his payoff [or playing against a non-strategic adversary, who only add uncertainty] Thus, we are making a distinctions between the situations when your opponents are strategic or non-strategic: For instance, you would like to spread a gossip (after witnessing a violent crime) to your friends, but only a small random subset of friends are on-line, as opposed to the situation when most of your friends have strategically chosen to be offline in order to avoid receiving this gossip from you.

Set of options or strategies:

$$S = \{s_1, s_2, \ldots, s_n\}.$$

Utility function – a real-valued function: $u(\cdot)$ gives a ranking among different options:

$$u : S \to \mathbb{R},$$

induces the ordering

$$u(s_{i_1}) \geq u(s_{i_2}) \geq \cdots \geq u(s_{i_n}).$$

We will need to extend this formalism to accommodate multi-player situation, as first proposed by John von Neumann and Oskar Morgenstern.

Expected Utility Theory: Starting from a set of "reasonable" axioms for *Rational Decision Making Under Uncertainty*.

Under uncertainty, every choice induces a "lottery," which yields probability distributions over different outcomes.

Theorem (Follows from vNM axioms)

There exists a utility function, called Bernoulli Utility function, $u(c)$, which gives utility of a consequence (outcome) c. Thus every choice a induces a probability distributions over consequences: $F^a(c)$, and the expected utility takes the value

$$\begin{aligned} U(a) & \\ &= \int u(c)dF^a(c) \\ &= \begin{cases} \int u(c)f^a(c)dc & \text{For cont. distribution with density } f^a(c) \\ \sum u(c_i)p_i^a & \text{For discrete distribution with prob. } p_i\text{'s} \end{cases} \end{aligned}$$

Under rationality, in a single-person game, if an individual has to choose among two actions a and b, with associated distributions $f^a(c)$ and $F^b(C)$, he will prefer a to b, iff $U(a) \geq U(b)$, where

$$U(a) = \int u(c)dF^a(c) \quad \text{and} \quad U(b) = \int u(c)dF^b(c).$$

Multi-player games are significantly more complex, as player 1's decision depends on player 2's, which in turn depends on player 1's, ad infinitum. Consider the game called "Battle of the Sexes (BoS)." The game involves two players: two friends of oppowite sexes: F (female) and M (male).

M / F	Opera	Football
Opera	3,2	0,0
Foorball	0,0	2,3

Table 1: Table for a Battle of the Sexes (BoS) game.

First number in the matrix entry is the pay-off to player 1 (row-player; female) and the second number is the payoff to player 2 (column-player; male).

Player 1 chooses a row: namely $x \in \{\text{opera}, \text{football}\}$ and player 2 chooses a column: namely $y \in \{\text{opera}, \text{football}\}$.

The payoffs are $a = u_1(x, y)$ and $b = u_2(x, y)$, and thus the entry in the matrix for the location (x, y) is (a, b).

Thus if F chooses opera and M chooses football then their payoffs are $0, 0$, since they will not be able to enjoy each other's company in this situation.

If, however, M (strategically) changes his choice to opera (even if he wouldn't go to opera just by himself), then the pay-offs increase to $3, 2$, since while F enjoys both the opera and M's company, M only gains some utility by being with F.

If both of them decide to be altruistic and make sacrifices for the other, then the situation is not necessarily better: F would then be choosing football, while M would be choosing opera, with the unfortunate situation of still not being able to benefit from each other's company; they end up with payoffs $0, 0$. (Perhaps, in this case, the payoffs could be modeled to be even worse: e.g., $-1, -1$. But, for the time being, the simplest model would do.)

The ideal situation would be if they can go to about equal numbers of (opera, opera) and (football, football), achieving payoffs averaging to $2\frac{1}{2}, 2\frac{1}{2}$.

F_2 / F_1	work hard	shirk
work hard	2,2	-1,1
shirk	1,-1	0,0

Table 2: Table for a Partnership game.

Another game (Partnership) game explores another situation, where both players would be better off, if they both work hard; but they are weary of the situation, when the other could take advantage of the situation by shirking. Many other real-life situations like these that we face daily can be modeled by games such as these.

Strategic Form Games (Normal Form Games or Matrix Games)

All participants act simultaneously and without knowledge of other players' actions.

Main ingredients: (i) the set of players, (ii) the strategies, and (iii) the payoffs.

In general, we may also need

- The Game Forms (which captures order of play)

- The Information Sets (which models asymmetric or incomplete information situation)

Formal Definition: A *strategic form game* is a triplet

$$\langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle$$

such that

- I is a finite set of players

$$I = \{1, 2, \ldots, l\};$$

- S_i (for $i \in I$) is the set of available actions (strategies) for player i
- $s_i \in S_i$ is an action for player i
-

$$u_i : S \to \mathbb{R}$$

is the payoff (utility) function of player i, where

$$S = \prod_i S_i$$

is the set of all action profiles.

Notation:

$$s_{-i} = [s_j]_{j \neq i},$$

vector of actions for all players except i.

$$S_{-i} = \prod_{j \neq i} S_i,$$

Set of all strategy profiles for all players except i.

$$(s_i, s_{-i}) \in S$$

is a strategy profile (or outcome).

Concept of Best Response:

$$B_i(S_{-i}) \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i});$$

The main question in game theory is whether everyone can choose "best responses" $s*$ such that

$$B_i(s*_{-i}) = s*_i \,.$$

There are some problems with this approach as shown by the two games: Matching Penny and Rock-Paper-Scissors.

Matcher \ Mismatcher	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Table 3: Table for a Matching-Penny game.

F1 \ F2	Rock	Paper	Scissors
Rock	0,0	-1,1	1, -1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Table 4: Table for a Rock-Paper-Scissors game.