$\boxed{\text{Lecture\#2}}$ 1

Boolean Connectives $\left\{\begin{array}{l}\text{E.g.}\\ \text{And } \wedge \\ \text{Or } \vee \\ \text{Implication } \Rightarrow \end{array}\right.$ Negation $\neg$ $\left.\right\}$ All can be derived from NAND

1. $A \wedge B$ is true iff A, B are both true and false otherwise

$$\wedge : \{0,1\}^2 \longrightarrow \{0,1\}$$

$$: (1,1) \mapsto 1 \qquad : (1,0) \mapsto 0$$

$$: (0,1) \mapsto 0 \qquad : (0,0) \mapsto 0$$

$A \cdot B$ multiplication over $\mathbb{Z}_2$

Value Matrix $\circ : \{0,1\}^2 \longrightarrow \{0,1\}$

$$\begin{pmatrix} 1\circ 1 & 1\circ 0 \\ 0\circ 1 & 0\circ 0 \end{pmatrix}$$ Truth Table

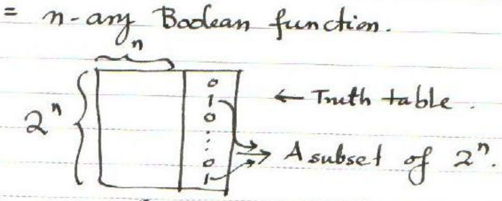
Truth Table / Value Matrix for $\wedge$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Boolean / Truth Function

A function $f : \{0,1\}^n \longrightarrow \{0,1\}$ is called an n-ary Boolean function or truth function.

$\mathbb{B}^n$ = n-ary Boolean function.



$|\mathbb{B}^n| = 2^{2^n}$

(2) Disjunction (Inclusive Or)

A or B $\quad A \vee B \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(3) Implication

If A then B $\quad A \Rightarrow B \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(B provided A) $\quad \updownarrow$

$[\neg A \vee B]$

(4) Equivalence

A iff B $\quad A \Leftrightarrow B \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5 Exclusive Disjunction (Parity)

A xor B $\qquad$ $A + B$ $\qquad$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(A+B) \bmod 2.$

6 Nihilation

Neither A nor B $\qquad$ $A \downarrow B$ $\qquad$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

A nor B

7 Incompatibility

Not at once A and B $\qquad$ $A \uparrow B$ $\qquad$ $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

A nand B

④

Language
Metalanguage &
Paradoxes

S = "This statement is false."

$S = 0 \Leftrightarrow S = 1.$

Liar's Paradox = "I am a liar."
Self-Reference.

Island of Knights and knaves where Knights always tell the truth and knaves always lie → "I am a knave"

Formalism
A Formal Language $\}$

Propositional Formula →
Strings of <u>symbols</u> built in given ways from basic symbols.

PV ≡ Propositional Variables.
→ Symbolized by $p_0, p_1, \ldots$ etc.

LC ≡ Logical Connectives
→ Symbolized by $\wedge, \vee, \neg, \Rightarrow, \ldots$ etc.

Parantheses
→ ( )

$\left.\begin{array}{l}\text{Well-Formed}\\ \text{Formulas}\\ \text{wff}\end{array}\right\}\mathcal{F}$

$$\mathcal{F} := p_0 \mid (\mathcal{F}_1 \wedge \mathcal{F}_2) \mid (\mathcal{F}_1 \vee \mathcal{F}_2) \mid \neg \mathcal{F} \; \cancel{\mid (\mathcal{F})}$$

$$(p_1 \wedge (p_2 \vee \neg p_1)) = \text{valid wff}$$

Propositional Language:

$\mathcal{F}$ of formulas built up from the symbols:

(Logical Signature)

$(\, , \,) \, , \, \wedge , \, \vee , \, \neg \, \ldots$ and

(logical variables)

$p_1, p_2, \ldots$

inductively as follows:

(F_1) The atomic strings $p_1, p_2, \cdots$ are formulas, called <u>prime formulas</u> [also called <u>atomic formulas</u>, or <u>primes</u>]

(F_2) If the strings α and β are formulas, then so too are strings

$(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $\neg\alpha$.

Stated Set-Theoretically:

$\mathcal{F}$ is the smallest (i.e. the intersection) of all sets of strings S built from the logical signatures and propositional variable symbols with the properties:

(f_1) $p_1, p_2, \cdots \in S$

(f_2) $\alpha, \beta \in S \Rightarrow (\alpha \wedge \beta), (\alpha \vee \beta), \neg\alpha \in S$

Boolean Formulas:

Obtained using Boolean signature: $\{\wedge, \vee, \neg\}$

Other Connectives.

$$\alpha \rightarrow \beta \equiv \neg(\alpha \wedge \neg\beta) \equiv \neg\alpha \vee \beta$$

$$\alpha \leftrightarrow \beta \equiv ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$$

Always True $\rightarrow$ Tautology, Verum, Top, $\top$

Aways False $\rightarrow$ Contradiction, Faloum, Bottom, $\bot$

$$\begin{tikzcd}[column sep=small] & \top \arrow[dl, dash] \arrow[dr, dash] & \\ T \arrow[dr, dash] & & F \arrow[dl, dash] \\ & \bot & \end{tikzcd}$$

Lattice

$$\top \equiv (T \vee F) \equiv (\alpha \vee \neg\alpha)$$

$$\bot \equiv (T \wedge F) \equiv (\alpha \wedge \neg\alpha)$$

$$\equiv \neg\bot$$

Law of Excluded Midde (LEM)

Notations

$p, q, \cdots$ $\equiv$ PV, propositional variables

$\alpha, \beta, \cdots$ $\equiv$ $\mathcal{F}$, formulas (wff)

π $\equiv$ PF, prime formulas

$X, Y, Z, \cdots$ $\equiv$ Propositional formulas PF.

Convention

(C_1) The outermost parantheses may be omitted

$$((\alpha \wedge \beta) \vee \neg \alpha) \equiv (\alpha \wedge \beta) \vee \neg \alpha$$

(C_2) In the order

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow,$

the former binds more strongly than the latter.

$$((\alpha \wedge \beta) \vee \neg \alpha) \equiv \alpha \wedge \beta \vee \neg \alpha$$

(C_3)

$\rightarrow$ ≡ Right Associative

$$\alpha \rightarrow \beta \rightarrow \gamma \equiv \alpha \rightarrow (\beta \rightarrow \gamma)$$

$\wedge, \vee$ ≡ Left Associative

$$\alpha \wedge \beta \wedge \gamma \equiv (\alpha \wedge \beta) \wedge \gamma$$

$$\alpha \vee \beta \vee \gamma \equiv (\alpha \vee \beta) \vee \gamma$$

INDUCTION.

On the construction of a formula
↓
Parse-Tree.

Principle of Formula Induction.

Let $\mathcal{E}$ be a property of strings that satisfy the following conditions:

(0) Base Case: $\mathcal{E}\pi$ for all prime formulas π

(S) Induction Case:

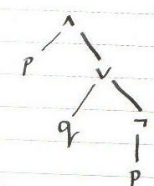
$\mathcal{E}\alpha, \mathcal{E}\beta \Rightarrow \mathcal{E}(\alpha \wedge \beta), \mathcal{E}(\alpha \vee \beta), \mathcal{E}\neg\alpha$
for all $\alpha, \beta \in \mathcal{F}$

Then $\mathcal{E}\varphi$ holds for all formulas (wffs) φ.

$\mathcal{E}\varphi \equiv$ Property $\mathcal{E}$ holding for string φ.

$$\frac{\mathcal{E}\pi;\ \mathcal{E}\alpha, \mathcal{E}\beta \Rightarrow \mathcal{E}(\alpha \wedge \beta), \mathcal{E}(\alpha \vee \beta)\ \mathcal{E}\neg\alpha}{\mathcal{E}\varphi}$$

$p \wedge (q \vee \neg p)$



Parse Tree.

Prefix Not^n ≡ Polish Normal Form (PN)

$\wedge p \vee q \neg p$ (PN)

Suffix Not^n ≡ Reverse Polish Normal Form (RPN)

$p\, q\, p \neg \vee \wedge$ (RPN)

PN → Polish Not^n (Prefix Not^n)

$\alpha, \beta \in \mathcal{F} \Rightarrow \neg\alpha,\ \wedge\alpha\beta,\ \vee\alpha\beta \in \mathcal{F}$.

RPN → Reverse Polish Not^n (Suffix Not^n)

$\alpha, \beta \in \mathcal{F} \Rightarrow \alpha\neg,\ \alpha\beta\wedge,\ \alpha\beta\vee \in \mathcal{F}$

Example (Inductive Def^n) $SF\varphi$ ≡ Subformula

$\forall \pi \in AF$ $\quad SF\pi \equiv \{\pi\}$

$SF\neg\alpha \equiv SF\alpha \cup \{\neg\alpha\}$

$SF(\alpha \circ \beta) \equiv SF\alpha \cup SF\beta \cup \{\alpha \circ \beta\}$

For a binary connective $\circ \in \{\wedge, \vee\}$

Example Rank $\quad rk\ \varphi$

$\equiv$ Highest Number of nested connectives in φ.

$$\forall_{\pi \in AF} \quad rk\ \pi \equiv 0$$
$$rk\ \neg\alpha \equiv 1 + rk\ \alpha$$
$$rk\ (\alpha \circ \beta) \equiv 1 + \max(rk\,\alpha, rk\,\beta)$$

$\circ \in \{\wedge, \vee\}$ Binary connectives.

Truth Value.

Truth value of a connected sentence depends only on the truth values of its constituent parts.

$$w : PV \to \{0,1\}$$

Extend the mapping from prime formulas to $\{0, 1\}$ $\longrightarrow$ to a mapping from the whole of $\mathcal{F}$ to $\{0,1\}$

$$\omega(\alpha \wedge \beta) = \omega\alpha \cdot \omega\beta$$
$$\omega(\alpha \vee \beta) = \cancel{\omega\alpha \oplus \omega\beta}\ \max(\omega\alpha, \omega\beta)$$
$$\omega\neg\alpha = 1 - \omega\alpha$$

Note $\quad \omega\top = \omega(\alpha \vee \neg\alpha) = \max(\omega\alpha, \omega\neg\alpha) = \max(\omega\alpha, 1-\omega\alpha) = 1$

$$\omega\bot = \omega(\alpha \wedge \neg\alpha) = \omega\alpha \cdot \omega\neg\alpha = \omega\alpha(1-\omega\alpha) = 0$$

NORMAL FORMS:

Semantic Equivalence

ω = Propositional Valuation

$\alpha \equiv \beta$ iff $\forall_{\text{valuation } \omega}$ $\omega\alpha = \omega\beta$.

Formulas α and β are $\left\{\begin{array}{l}\text{logically} \\ \text{semantically}\end{array}\right\}$ equivalent.

① $\alpha \equiv \neg\neg\alpha$

② Associativity: $\alpha \wedge (\beta \wedge \gamma) \equiv \alpha \wedge \beta \wedge \gamma$;
$\alpha \vee (\beta \vee \gamma) \equiv \alpha \vee \beta \vee \gamma$

③ Commutativity: $\alpha \wedge \beta \equiv \beta \wedge \alpha$; $\alpha \vee \beta \equiv \beta \vee \alpha$

④ Idempotent: $\alpha \wedge \alpha \equiv \alpha$; $\alpha \vee \alpha \equiv \alpha$

⑤ Absorption: $\alpha \wedge (\alpha \wedge \beta) \equiv \alpha$; $\alpha \vee (\alpha \wedge \beta) \equiv \alpha$

⑥ $\wedge$- Distributivity: $\alpha \wedge (\beta \vee \gamma) \equiv \alpha \wedge \beta \vee \alpha \wedge \gamma$

⑦ $\vee$ - Distributivity: $\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

⑧ de Morgan Rules:

$\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$; $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$

⑨ $\alpha \vee \neg\alpha \equiv \top$ $\quad$ $\alpha \wedge \neg\alpha \equiv \bot$ $\quad$ $\alpha \wedge \top \equiv \alpha \vee \bot \equiv \alpha$