Some of these problems could be quite difficult and do not necessarily represent the difficulty level of the final exam.

**Problem**

The Fibonacci sequence \( \{F_k \mid k = 0, 1, 2, \ldots\} \) is defined as follows:

\[
F_0 = 0, F_1 = 1, \quad \text{and} \quad \forall k \geq 2, \quad F_k = F_{k-1} + F_{k-2}.
\]

1. Prove that \( \forall k \geq 2, \quad 2F_k = F_{k+1} + F_{k-2} \).

2. Let \( \phi > 0, \ psi < 0 \) be the roots of the quadratic equation \( 1 + x = x^2 \). Prove that

\[
\forall k \geq 0, \quad F_k = \frac{1}{\sqrt{5}} (\phi^k - \psi^k) = \left\lfloor \frac{\phi^k}{\sqrt{5}} \right\rfloor
\]

where \( \lfloor z \rfloor \) denotes the nearest integer to \( z \).

3. We want to express a given positive integer \( n \) as a sum of Fibonacci numbers using the minimum number of terms. Consider a recursive greedy algorithm that finds the largest index \( k \) such that \( F_k \leq n \), writes \( n = F_k + n' \) and then recursively expresses \( n' \) as a sum of Fibonacci numbers (the algorithm stops if \( n' = 0 \)). Prove that this algorithm finds an expression for a given positive integer as a sum of Fibonacci numbers using the minimum number of terms.

**Problem**

Given a set of intervals on a line, design a polynomial time greedy algorithm to select minimum number of intervals such that every interval overlaps with at least one of the selected intervals.

**Problem**

You are given \( k \) sorted lists with \( n \) elements each. Design an algorithm, as efficient as possible, to merge all of them together into a single sorted list.

**Problem**

Suppose there is a collection of \( n \) intervals \( \{[a_i, b_i] \mid i = 1, \ldots, n\} \) such that every two intervals in this collection overlap. Prove that all the intervals in the collection overlap, i.e. there is a point on the real line that belongs to all these intervals.
Problem
Given a sequence of distinct integers \((a_1, a_2, \ldots, a_n)\), design an \(O(n)\) time algorithm to find for each number \(a_i\) the smallest \(j\) such that \(a_j > a_i\) and \(i < j \leq n\), if such a \(j\) exists. That is, for each \(a_i\), find the first integer to its right which is larger than it.

Problem
Consider a chess-board of size \(2^k \times 2^k\) and remove any one its squares. Prove that the remaining board can be tiled with \(L\)-shaped tiles of size 3 (i.e. a tile with 3 squares arranged in \(L\)-shape). Use induction on \(k\) and a strategy based on Divide and Conquer paradigm.

Problem
Given a sequence of integers \((a_1, a_2, \ldots, a_n)\), each integer either positive or negative, give an \(O(n \log n)\) time algorithm to find a shortest sub-sequence of consecutive integers \(a_i, a_{i+1}, \ldots, a_j\) whose sum is at least a given integer \(M\).