

G22.3520: Honors Analysis of Algorithms

Practice Problems

Some of these problems could be quite difficult and do not necessarily represent the difficulty level of the final exam.

Problem

The Fibonacci sequence $\{F_k \mid k = 0, 1, 2, \dots\}$ is defined as follows:

$$F_0 = 0, F_1 = 1, \quad \text{and} \quad \forall k \geq 2, \quad F_k = F_{k-1} + F_{k-2}.$$

1. Prove that $\forall k \geq 2, \quad 2F_k = F_{k+1} + F_{k-2}$.
2. Let $\phi > 0, \psi < 0$ be the roots of the quadratic equation $1 + x = x^2$. Prove that

$$\forall k \geq 0, \quad F_k = \frac{1}{\sqrt{5}}(\phi^k - \psi^k) = \lfloor \frac{\phi^k}{\sqrt{5}} \rfloor$$

where $\lfloor z \rfloor$ denotes the nearest integer to z .

3. We want to express a given positive integer n as a sum of Fibonacci numbers using the minimum number of terms. Consider a recursive greedy algorithm that finds the largest index k such that $F_k \leq n$, writes $n = F_k + n'$ and then recursively expresses n' as a sum of Fibonacci numbers (the algorithm stops if $n' = 0$). Prove that this algorithm finds an expression for a given positive integer as a sum of Fibonacci numbers using the minimum number of terms.

Problem

Given a set of intervals on a line, design a polynomial time greedy algorithm to select minimum number of intervals such that every interval overlaps with at least one of the selected intervals.

Problem

You are given k sorted lists with n elements each. Design an algorithm, as efficient as possible, to merge all of them together into a single sorted list.

Problem

Suppose there is a collection of n intervals $\{[a_i, b_i] \mid i = 1, \dots, n\}$ such that every two intervals in this collection overlap. Prove that all the intervals in the collection overlap, i.e. there is a point on the real line that belongs to all these intervals.

Problem

Given a sequence of distinct integers (a_1, a_2, \dots, a_n) , design an $O(n)$ time algorithm to find for each number a_i the smallest j such that $a_j > a_i$ and $i < j \leq n$, if such a j exists. That is, for each a_i , find the first integer to its right which is larger than it.

Problem

Consider a chess-board of size $2^k \times 2^k$ and remove any one of its squares. Prove that the remaining board can be *tiled* with L -shaped tiles of size 3 (i.e. a tile with 3 squares arranged in L -shape). Use induction on k and a strategy based on Divide and Conquer paradigm.

Problem

Given a sequence of integers (a_1, a_2, \dots, a_n) , each integer either positive or negative, give an $O(n \log n)$ time algorithm to find a shortest sub-sequence of consecutive integers a_i, a_{i+1}, \dots, a_j whose sum is at least a given integer M .