Homework 3
Computational Complexity

April 11, 2010

Due on Monday, May 3. Collaboration is allowed; please mention your collaborators.

1. The class MA is analogous to NP where the verifier can be a randomized algorithm. There is an all powerful prover (called Merlin) who gives a proof to the probabilistic polynomial time verifier (called Arthur). Arthur uses a (private) random string r.
   Define the class of languages $\mathrm{MA}_{2/3,1/3}$ with two sided error as follows.

   $$\begin{aligned} x \in L &\Rightarrow \exists y, \Pr_r[V(x,y,r)=1] \geq \frac{2}{3} \\ x \notin L &\Rightarrow \forall y, \Pr_r[V(x,y,r)=1] \leq \frac{1}{3} \end{aligned}$$

   Here $V(,)$ is a deterministic polynomial time verification procedure and lengths of y are r are polynomially bounded in the length of x. Similarly we define the class $\mathrm{MA}_{1,1/3}$ with one-sided error as follows

   $$\begin{aligned} x \in L &\Rightarrow \exists y, \Pr_r[V(x,y,r)=1] = 1 \\ x \notin L &\Rightarrow \forall y, \Pr_r[V(x,y,r)=1] \leq \frac{1}{3} \end{aligned}$$

   Show that $\mathrm{MA}_{2/3,1/3} = \mathrm{MA}_{1,1/3}$. That is, if a language has a MA-protocol with two-sided error, then it also has a MA-protocol with one-sided error. *Hint: Use ideas from the proof of* $\mathrm{BPP} \subseteq \Sigma_2$.

2. Show that

   $$\mathrm{PSPACE} \subseteq \mathrm{P/poly} \quad \Rightarrow \quad \mathrm{PSPACE} = \Sigma_2$$

   Hint: Modify the proof of Karp-Lipton Theorem for a self reducible PSPACE complete problem.

3. In this question all circuit classes are non-uniform. Show that for any non-negative integer i,

$$NC^i = NC^{i+1} \quad \Rightarrow \quad NC = NC^i$$

4. Assume that the problem of counting the number of matchings (not just perfect matchings) in a graph is #P-complete. Show that the problem of counting the number of satisfying assignments to an instance of 2-SAT is #P-complete.

5. **Pairwise Independent Hash Functions**
   Consider the following family of functions F that map $\{0,1\}^n \rightarrow \{0,1\}^k$. Pick a $k \times n$ matrix A with $0,1$ entries at random. Pick $b \in \{0,1\}^k$ at random. Let

$$f(x) = Ax + b$$

   where all arithmetic operations are over Z_2. Assume that $f \in F$ is picked uniformly at random (by choosing A and b randomly).

   - Show that for any $x \in \{0,1\}^n$ and $y \in \{0,1\}^k$,

$$Pr_{A,b}[f(x) = y] = \frac{1}{2^k}$$

     Hint: first consider the case when $k = 1$

   - Show that for any $x_1, x_2 \in \{0,1\}^n$ and $x_1 \neq x_2$, and any $y_1, y_2 \in \{0,1\}^k$,

$$Pr_{A,b}[(f(x_1) = y_1) \wedge (f(x_2) = y_2)] = \frac{1}{2^{2k}}$$

   - Show that for any $x_1, x_2 \in \{0,1\}^n$ and $x_1 \neq x_2$,

$$Pr_{A,b}[f(x_1) = f(x_2)] = \frac{1}{2^k}$$

6. We will use pairwise independent hash functions to design an AM protocol for MANY-SAT. The problem is that we are given a SAT instance ϕ with S as the set of its satisfying assignments. We are told that either $|S| \geq 2^k$ (YES case) or $|S| \leq 2^{k-10}$ (NO case). We have to distinguish the YES and NO cases.
   Consider the following AM protocol for MANY-SAT. Arthur picks a

random hash function $f_{A,b} : \{0,1\}^n \to \{0,1\}^k$, and a random target value $y \in \{0,1\}^k$. Merlin sends $x \in \{0,1\}^n$ as answer. Arthur accepts iff $f_{A,b}(x) = y$ and x is a satisfying assignment to ϕ.
Show that this is a valid AM protocol i.e the probability of acceptance in the YES case is significantly larger than the NO case.
Hint: In the YES case, show that there are not too many collisions, the size of the image of S, i.e. $|f_{A,b}(S)|$ is likely to be large, and a random $y \in \{0,1\}^k$ is likely to have a pre-image.