Honors Analysis of Algorithms
Problem Set 5

Collaboration is allowed, but you must write your own solutions. Proofs of correctness are a must.

Problem 1 (Randomized Algorithms)
Solve: [Kleinberg Tardos] Chapter 13, problem 1, page 782.

Problem 2 (Randomized Algorithms)
Hint: Consider an agent such that there are \( k \) agents with a higher bid than her. What is the probability that her bid results in an update of \( b^* \)?

Problem 3 (Randomized Algorithms)

Problem 4
Solve Problem 4 from: http://www.cs.nyu.edu/courses/fall07/G22.3520-001/ps2.pdf
Note: This is the problem about the algorithm \( A(S) \). You can assume that \( R \) is chosen such that \( R \neq \emptyset \) and \( R \neq S \) (so that the recursive calls terminate in at most \( |S| \) steps). What is an upper bound on a value output by the algorithm? You can first solve the problem assuming that when a random subset \( R \subseteq S \) is chosen, it is always the case that \( \frac{|S|}{3} \leq |R| \leq \frac{2|S|}{3} \). Now you can relax the assumption noting that the probability that \( |R| \) is outside of this range is at most \( 2^{-\alpha|S|} \) for some universal constant \( \alpha \).

Problem 5
In this problem, we explore the notion of oracle reducibility. If \( A \) is a language, then a Turing machine with oracle \( A \) is a Turing machine with a “magical” subroutine that decides membership in \( A \). In other words, the subroutine, when given a string \( w \), tells the machine whether or not \( w \in A \). Let
\[
\text{HALT}_{TM} = \{(M,x) \mid M \text{ is a Turing machine that halts on } x\}.
\]
Show that there is a Turing machine with oracle \( \text{HALT}_{TM} \) that decides the following problem with only two questions to the oracle: Given three (machine, input) pairs \((M_1,x_1), (M_2,x_2), (M_3,x_3)\), decide for each pair whether the Turing machine halts on the corresponding input.
Note: This is trivial if one allows three questions. Just ask the oracle whether $\langle M_i, x_i \rangle \in \text{HALT}_{TM}$ for $i = 1, 2, 3$.

**Problem 6**

Show that the collection of Turing-recognizable languages is closed under the operation of (a) union (b) concatenation (c) star and (d) intersection. What about complementation operation?