GA.3520: Honors Analysis of Algorithms

Practice Problems

Some of these problems could be quite difficult and do not necessarily represent the difficulty level of the final exam.

Problem

The Fibonacci sequence \( \{F_k \mid k = 0, 1, 2, \ldots\} \) is defined as follows:

\[
F_0 = 0, F_1 = 1, \quad \text{and} \quad \forall \ k \geq 2, \quad F_k = F_{k-1} + F_{k-2}.
\]

1. Prove that \( \forall \ k \geq 2, \quad 2F_k = F_{k+1} + F_{k-2} \).

2. Let \( \phi > 0, \ \psi < 0 \) be the roots of the quadratic equation \( 1 + x = x^2 \). Prove that

\[
\forall \ k \geq 0, \quad F_k = \frac{1}{\sqrt{5}}(\phi^k - \psi^k) = \left\lfloor \frac{\phi^k}{\sqrt{5}} \right\rfloor
\]

where \( \lfloor z \rfloor \) denotes the nearest integer to \( z \).

3. We want to express a given positive integer \( n \) as a sum of Fibonacci numbers using the minimum number of terms. Consider a recursive greedy algorithm that finds the largest index \( k \) such that \( F_k \leq n \), writes \( n = F_k + n' \) and then recursively expresses \( n' \) as a sum of Fibonacci numbers (the algorithm stops if \( n' = 0 \)). Prove that this algorithm finds an expression for a given positive integer as a sum of Fibonacci numbers using the minimum number of terms.

Problem

Given a set of intervals on a line, design a polynomial time greedy algorithm to select minimum number of intervals such that every interval overlaps with at least one of the selected intervals.

Problem

You are given \( k \) sorted lists with \( n \) elements each. Design an algorithm, as efficient as possible, to merge all of them together into a single sorted list.

Problem

Suppose there is a collection of \( n \) intervals \( \{[a_i, b_i] \mid i = 1, \ldots, n\} \) such that every two intervals in this collection overlap. Prove that all the intervals in the collection overlap, i.e. there is a point on the real line that belongs to all these intervals.
Problem

Given a sequence of distinct integers \((a_1, a_2, \ldots, a_n)\), design an \(O(n)\) time algorithm to find for each number \(a_i\) the smallest \(j\) such that \(a_j > a_i\) and \(i < j \leq n\), if such a \(j\) exists. That is, for each \(a_i\), find the first integer to its right which is larger than it.

Problem

Consider a chess-board of size \(2^k \times 2^k\) and remove any one its squares. Prove that the remaining board can be tiled with \(L\)-shaped tiles of size 3 (i.e. a tile with 3 squares arranged in \(L\)-shape). Use induction on \(k\) and a strategy based on Divide and Conquer paradigm.

Problem

Given a sequence of integers \((a_1, a_2, \ldots, a_n)\), each integer either positive or negative, give an \(O(n \log n)\) time algorithm to find a shortest sub-sequence of consecutive integers \(a_i, a_{i+1}, \ldots, a_j\) whose sum is at least a given integer \(M\).