You are expected to solve all the problems. Due on Wed, Mar 12th. Collaboration is allowed; please mention your collaborators.

1. Show that $\Sigma_k = \text{NP}^{\text{CIRCUIT-SAT}_{k-1}}$.

We define $L \in \Sigma_k$ if and only if there is a deterministic polynomial time verifier $V$ such that

$$x \in L \iff \exists y_1 \forall y_2 \cdots Q_k y_k \ V(x, y_1, \cdots, y_k) = 1$$

where length of $y_1, y_2, \ldots, y_k$ is bounded by a polynomial in $|x|$.

2. • Show that $\text{P}^{\text{PSPACE}} = \text{NP}^{\text{PSPACE}} = \text{PSPACE}$.
   • Show that if $\text{PH} = \text{PSPACE}$, then $\text{PH}$ collapses to some finite level.
   • Can $\text{PH}$ have a complete problem (complete under polynomial time reductions)?

3. (DP-completeness) This problem studies the class DP (D stands for difference). A language $L \in \text{DP}$ if and only if there are languages $B \in \text{NP}$ and $C \in \text{coNP}$ so that $L = B \cap C$.

   • The problem SAT-UNSAT is defined as follows: Given pair of Boolean formulae ($\phi, \psi$), decide if $\phi$ is satisfiable and $\psi$ is unsatisfiable. Show that this problem is DP-complete (under polynomial time reductions).
   • A graph $G$ is in HC-CRITICAL is $G$ is not Hamiltonian but adding any edge to $G$ will make it Hamiltonian. Show that HC-CRITICAL is in DP.
4. • Show that $\text{NP}^\text{BPP} \subseteq \text{BPP}^\text{NP}$ (Hint: First show that a language in $\text{NP}^\text{BPP}$ is accepted by a polytime NTM that makes a single query to a BPP oracle and that too at the end).
  
  • Show that if $\text{NP} \subseteq \text{BPP}$, then PH collapses to BPP.

5. (NEXP-completeness) Define $\text{NEXP} = \bigcup_{k=1,2,...} \text{NTIME}(2^{n^k})$.
   Show that the following problem is NEXP-complete: Given $\langle M, x, n \rangle$, consisting of description of a NTM $M$, input $x$ and an integer $n$ in binary, decide if $M$ has an accepting computation on $x$ in $n$ steps.

6. A circuit $C$ is called an implicit representation of another circuit $C^*$ if $C$ takes as input a binary integer $i$ such that $n + 1 \leq i \leq N$, and outputs a triple $(\text{TYPE}, j, k)$ where

   • Input to $C^*$ is an $n$-bit string $x_1x_2 \ldots x_n$.
   • $\text{TYPE} \in \{\text{AND, OR, NOT}\}$ indicates the type of $i^{th}$ gate in circuit $C^*$.
   • $1 \leq j, k \leq N$.
   • The input of the $i^{th}$ gate in $C^*$ is the output of the $j^{th}$ and $k^{th}$ gates of $C^*$ (if TYPE= NOT, then $k$ is ignored. If $1 \leq j, k \leq n$, then the $j^{th}$ or $k^{th}$ gate is taken to be an input bit $x_i$).
   • The $N^{th}$ gate in $C^*$ is its output gate.

   Note that we could have $N = 2^n$, the circuit $C$ could be of size poly$(\log N) = \text{poly}(n)$ and still implicitly represent a circuit $C^*$ of size $N$ (in short, a circuit can implicitly represent another circuit of size exponential in its own size).

   Let IMPLICIT CIRCUIT-SAT be the following problem: Given a circuit $C$ that is an implicit representation of circuit $C^*$, decide if $C^*$ is satisfiable. Show that this problem is NEXP-complete (Hint: Use the regular structure of the circuit produced in Cook-Levin reduction).

7. • Show that $\text{NP}^{\text{NP} \cap \text{coNP}} = \text{NP}$.
  
  • Generalize this to $\text{NP}^{\Sigma_k \cap \Pi_k} = \Sigma_k$.

8. The problem Graph Consistency (GC) asks, for two given sets $A$ and $B$ of graphs, whether there exists a graph $G$ such that every graph $g \in A$ is isomorphic to a (not necessarily induced) subgraph of $G$ but each graph $h \in B$ is not isomorphic to any subgraph of $G$. Show that GC is in $\Sigma_2$. 
9. Show that if $\Sigma_k = \Pi_k$ for some $k$, then $\text{PH} = \Sigma_k$. 