A Note on Lower Bounds for Non-interactive Message Authentication Using Weak Keys

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Abstract

In this note, we prove lower bounds on the amount of entropy of random sources necessary for secure message authentication. We consider the problem of non-interactive $c$-time message authentication using a weak secret key having min-entropy $k$. We show that existing constructions using $(c + 1)$-wise independent hash functions are optimal.

This result resolves one of the main questions left open by the work of Dodis and Spencer [2] who considered this problem for one-time message authentication of one-bit messages.

1 Introduction

1.1 Non-interactive Message Authentication

In this note, we revisit the problem of non-interactive message authentication: where Alice and Bob share a weak secret key $R \in \{0,1\}^n$, and Alice wants to communicate up to $c$ messages authentically to Bob over a channel controlled by the adversary Eve. This problem is known to have an easy solution with $\varepsilon$-security for $\varepsilon < 1$ using one of various possible universal hash functions, or more generally $c + 1$-wise independent hash functions (see, for example, [6, 5] that give construction for $c = 1$). These solutions, however, require that the min-entropy $H_\infty(R)$ of the source $R$ is at least $\frac{cn}{c+1} + \log(\frac{1}{\varepsilon})$.

Dodis and Spencer [2] studied this problem with the goal of finding a lower bound on the min-entropy of $R$. They showed that for any integer $k \geq \frac{n}{2}$, and any one-round message authentication protocol for one-bit messages, there exists a $k$-flat source $R$ such that the advantage of the adversary in forging the tag is at least $2^{n/2-k}$, or in other words, $H_\infty(R) \geq \frac{n}{2} + \log(\frac{1}{\varepsilon})$. This showed that the construction using universal hash functions is optimal for one-bit messages. However, the bound for many time message authentication is still far from optimal and this was left as one of the main open questions in [2]. Specifically, the authors state that it is interesting to extend their quantitative results for private-key encryption and especially authentication to larger than one-bit message spaces. While this question has subsequently been almost resolved for the case of private-key encryption [1], it has remained open for the case of private-key authentication.

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1.2 Our contribution and Comparison with [2]

We answer this open question in the affirmative, i.e., that for any integer $k \geq \frac{cn}{c+1}$, and any $c$-round message authentication protocol, there exists a $k$-flat source $R$ such that the advantage of the adversary in forging the tag is at least $2^{cn/(c+1)-k}$, or in other words, $H_\infty(R) \geq \frac{cn}{c+1} + \log\left(\frac{1}{\varepsilon}\right)$. Our proof uses a simple idea based on the chain rule for Shannon entropy.

In comparison, the result of [2] was proved by considering a bipartite multigraph with the edges corresponding to the keys and the vertices on each part corresponding to the tags of the bit 0 and 1, respectively. They then partitioned their proof into two cases (i) where there are few tags corresponding to the bit 0, in which case it is easy to guess $\text{Tag}(0, R)$, and (ii) where there are many tags corresponding to the bit 0, but where knowing $\text{Tag}(0, R)$ gives significant information about $\text{Tag}(1, R)$. It seems that one might be able to generalize this idea to prove a lower bound for $c$-time message authentication by considering $c+1$ cases as opposed to considering two cases for $c = 1$. However, the case analysis becomes significantly more involved due to the combinatorial nature of the proof, and perhaps this is a reason why the question has remained open for so long.

2 Preliminaries

For a set $S$, we let $U_S$ denote the uniform distribution over $S$. For an integer $m \in \mathbb{N}$, we let $U_m$ denote the uniform distribution over $\{0, 1\}^m$, the bit-strings of length $m$. For a distribution or random variable $X$ we write $x \leftarrow X$ to denote the operation of sampling a random $x$ according to $X$. For a set $S$, we write $s \leftarrow S$ as shorthand for $s \leftarrow U_S$.

2.1 Entropy Definitions

The prediction probability of a random variable $X$ is defined as

$$\text{Pred}(X) := \max_x \Pr[X = x].$$

The min-entropy of $X$ is defined as

$$H_\infty(X) := -\log \text{Pred}(X).$$

We say that a random variable $X$ is an $(n, k)$-source if $X \in \{0, 1\}^n$ and $H_\infty(X) \geq k$. We also define conditional prediction probability of a random variable $X$ conditioned on another random variable $Z$ as

$$\text{Pred}(X|Z) := \mathbb{E}_{z \leftarrow Z} \left[ \max_x \Pr[X = x|Z = z] \right]$$

$$= \mathbb{E}_{z \leftarrow Z} \left[ 2^{-H_\infty(X|Z = z)} \right].$$

The conditional min-entropy of $X$ is defined as

$$H_\infty(X|Z) := -\log \text{Pred}(X|Z).$$

Also, the Shannon entropy $H_1(X)$ of a random variable $X$ is defined as

$$H_1(X) := -\sum_x \Pr[X = x] \log \Pr[X = x].$$
The conditional Shannon entropy of a random variable $X$ conditioned on another random variable $Z$ is defined as

$$H_1(X|Z) := E_{z \leftarrow Z} H_1(X|Z = z) = -E_{z \leftarrow Z} \sum_x \Pr[X = x|Z = z] \log \Pr[X = x|Z = z].$$

We will need the following standard facts about (conditional) min-entropy, and (conditional) Shannon entropy.

**Fact 1.** Let $X,Y,Z$ be arbitrary random variables, and let $f$ be an arbitrary function. Then the following hold

1. $H_{\infty}(X|Z) \geq H_{\infty}(f(X)|Z)$, and $H_1(X|Z) \geq H_1(f(X)|Z)$.
2. $H_1(X,Y|Z) = H_1(X|Y,Z) + H_1(Y|Z)$.
3. $H_1(X|Z) \geq H_{\infty}(X|Z)$.

We remark here that the definition of the conditional Shannon entropy is fairly standard, but there are other alternative definitions in the literature for conditional min-entropy. However, our proposed definition is by now fairly standard. We direct the reader to [3] which contains a comprehensive discussion on conditional entropies, and proves Fact 1 among several other results.

### 2.2 Message Authentication Codes

In order to define a message authentication code, we first introduce the following game $G_c(r)$. For a given function $Tag : \mathcal{M} \times \{0,1\}^n \mapsto \mathcal{T}$ and a fixed secret key $r \in \{0,1\}^n$, an adversary Eve is allowed to make at most $c$ adaptive queries $\mu_1, \ldots, \mu_c$ to $Tag(\cdot,r)$. We say that Eve wins the game if she outputs a pair $(\mu_{c+1}, \sigma)$, such that $Tag(\mu_{c+1}, r) = \sigma$ and $\mu_{c+1} \notin \{\mu_1, \ldots, \mu_c\}$. We define the advantage of Eve in this game as

$$Adv_Eve^c(r) = \Pr[Eve \text{ wins } G_c(r)].$$

**Definition 1.** A function $Tag : \mathcal{M} \times \{0,1\}^n \mapsto \mathcal{T}$ is called a $c$-time $(n,k,\varepsilon)$-secure message authentication code, if for any distribution $R$ on $\{0,1\}^n$ with $H_{\infty}(R) \geq k$, for any computationally unbounded adversary Eve,

$$E_{r \leftarrow R}[Adv_Eve^c(r)] \leq \varepsilon.

### 2.3 $k$-wise Independent Hash Functions

Here we define and give a well-known construction of $k$-wise independent hash functions.

**Definition 2.** A function $H : \mathcal{X} \times \mathcal{R} \mapsto \mathcal{Y}$ is said to be a $k$-wise independent hash function if for all $y_1, \ldots, y_k \in \mathcal{Y}$, and all distinct $x_1, \ldots, x_k \in \mathcal{X}$,

$$\Pr_{r \leftarrow \mathcal{R}}[H(x_1, r) = y_1 \land \cdots \land h(x_k, r) = y_k] = \frac{1}{|\mathcal{Y}|^k}.$$

**Lemma 1** (folklore). Let $k$ be a positive integer, and let $\mathcal{X} = \mathcal{Y} = \mathbb{F}$, and $\mathcal{R} = \mathbb{F}^k$ for some finite field $\mathbb{F}$. Then the function $H : \mathcal{X} \times \mathcal{R} \mapsto \mathcal{Y}$ given by

$$H(x, (r_0, \ldots, r_{k-1})) := r_0 + r_1 \cdot x + \cdots + r_{k-1} \cdot x^{k-1}$$

is a $k$-wise independent hash function.
3 Tight Bound for $c$-time MACs

In this section, we prove a lower bound on the error-probability $\varepsilon$ for $c$-time message authentication protocol for deterministic functions $\text{Tag}$.

**Theorem 1.** Let $\text{Tag}$ be a $c$-time $(n, k, \varepsilon)$-secure message authentication code where $\text{Tag} : \mathcal{M} \times \{0, 1\}^n \rightarrow \mathcal{T}$. Then we have the following.

1. If $k \leq \frac{c}{c+1} n$ then $\varepsilon = 1$;
2. If $k > \frac{c}{c+1} n$ then $\varepsilon \geq 2\frac{c}{c+1} n - k$.

**Proof.** Let $U$ be an $n$-bit uniformly random string, and let $\mu_1, \ldots, \mu_{c+1} \in \mathcal{M}$ be fixed distinct messages. Note that $H_1(U) = n$. Using Fact 1 multiple times, we get

$$
\sum_{i=1}^{c+1} H_1(\text{Tag}(\mu_i, U)) = \frac{n}{c+1} \leq \frac{n}{c+1} .
$$

Therefore, there exists $i \in \{1, \ldots, c+1\}$, such that

$$
H_\infty(\text{Tag}(\mu_i, U) | \text{Tag}(\mu_1, U), \ldots, \text{Tag}(\mu_{i-1}, U)) \leq \frac{n}{c+1} .
$$

We fix an $i$ satisfying this inequality. For any $t = (t_1, \ldots, t_{i-1}) \in \mathcal{T}^{i-1}$, let $\mathcal{E}(t)$ be a shorthand for the event that $\text{Tag}(\mu_j, U) = t_j$ for $1 \leq j < i$. From the definition of conditional min-entropy, we get the following.

$$
2^{-\frac{n}{c+1}} \leq \max_{t \in \mathcal{T}^{i-1}} \sum_{t_j \in \mathcal{T}} \Pr[\text{Tag}(\mu_j, U) = t_j | \mathcal{E}(t)]
$$

$$
= \sum_{t \in \mathcal{T}^{i-1}} \Pr[\mathcal{E}(t)] \cdot \max_{t_j \in \mathcal{T}} \Pr[\text{Tag}(\mu_j, U) = t_j | \mathcal{E}(t)]
$$

$$
= \sum_{t \in \mathcal{T}^{i-1}} \max_{t_j \in \mathcal{T}} \Pr[\text{Tag}(\mu_j, U) = t_j \text{ for } 1 \leq j \leq i] . \quad (1)
$$

For every fixed $t = (t_1, \ldots, t_{i-1}) \in \mathcal{T}^{i-1}$, let $\mu_t$ be the most probable value of $\text{Tag}(\mu_i, U)$ given $\text{Tag}(\mu_j, U) = t_j$ for $1 \leq j < i$. Intuitively, we want to choose a distribution over the set of keys so that $\text{Tag}(\mu_j, U) = t_j$ for $1 \leq j < i$ implies that $\text{Tag}(\mu_i, U) = \mu_t$. Then, given tags for $\mu_1, \ldots, \mu_{i-1}$, we can always guess the tag for $\mu_i$. Let $K_t$ be the set of keys corresponding to $\mu_t$, i.e.,

$$
K_t = \{ r \in \{0, 1\}^n | \text{Tag}(\mu_i, r) = \mu_t, \text{Tag}(\mu_j, r) = t_j \text{ for } 1 \leq j < i \} .
$$

Let also

$$
\mathcal{K} = \bigcup_{t \in \mathcal{T}^{i-1}} K_t .
$$

From inequality (1),

$$|K| \geq 2^n \cdot 2^{-\frac{n}{c+1}} = 2^{\frac{cn}{c+1}}.$$ 

If $2^k \leq |K|$, then let $R$ be an arbitrary $2^k$ element subset of $K$. Otherwise, let

$$R = K \cup K',$$

where $K'$ is a set of arbitrary keys from the set $\{0, 1\}^n \setminus K$, such that $|R| = 2^k$.

We claim that if $R$ is uniformly distributed on $R$, then there exists a strategy for Eve such that the advantage in guessing $Tag(\mu, r)$ given $Tag(\mu_1, r), \ldots, Tag(\mu_{i-1}, r)$ is at least $2^{\frac{cn}{c+1} - k}$ if $k > \frac{cn}{n+1}$, and 1, otherwise. To see this, notice that for any $r \in K$, there is a unique value of $Tag(\mu, r)$ given $Tag(\mu_1, r), \ldots, Tag(\mu_{i-1}, r)$. Let the strategy of Eve be to guess this unique tag assuming $R \in K$.

Then, Eve succeeds with probability $1$ if $R \in K$, and hence the advantage of Eve is

$$\varepsilon \geq \frac{|R \cap K|}{2^k} \geq \min \left( \frac{2^k \cdot 2^{\frac{cn}{c+1}}}{2^k} \right).$$

The statement of the theorem now follows.

It is well-known that the bound from Theorem 1 can be achieved by using a family of $(c+1)$-wise independent hash functions (see [4] for similar results). For the sake of completeness, we present this construction below.

**Lemma 2 (folklore).** Let $F$ be a finite field, and let $M = T = F$, and let the set of keys be $F^{c+1}$ with $n = (c+1) \log |F|$. Then the function $Tag : M \times F^{c+1} \mapsto T$ defined as:

$$Tag(\mu, (r_0, \ldots, r_c)) := r_0 + r_1 \cdot \mu + \cdots + r_c \cdot \mu^c$$

is a $c$-time $(n, k, 2^{\frac{cn}{c+1} - k})$-secure message authentication code.

**Proof.** Let $U$ be uniform in $F^{c+1}$. For any fixed strategy of Eve, and $r \in F^{c+1}$, let $f(r)$ denote $Adv^Eve(r)$. Let $\mu_1, \ldots, \mu_{c+1}$ be arbitrary distinct messages in $M$. By Lemma 1, we have that for any $\sigma \in T$, the probability that $Tag(\mu_{c+1}, U) = \sigma$ given $Tag(\mu_1, U), \ldots, Tag(\mu_c, U)$ is at most $\frac{1}{|F|} = 2^{-n/(c+1)}$. Hence,

$$E_{r \leftarrow U}[f(r)] \leq 2^{-\frac{n}{c+1}}.$$

Now, consider a random key $R \in F^{c+1}$, such that $H_\infty(R) \geq k$. Then

$$E_{r \leftarrow R}[f(r)] = \sum_{r \in F^{c+1}} \Pr(R = r) \cdot f(r)$$

$$\leq \max_{r \in F^{c+1}} \Pr(R = r) \sum_{r \in F^{c+1}} f(r)$$

$$\leq 2^{-k} \cdot 2^n \cdot E_{r \leftarrow U}[f(r)]$$

$$\leq 2^{-n - k} \cdot 2^{-\frac{n}{c+1}}$$

$$= 2^{\frac{cn}{c+1} - k},$$

as needed.
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References


