Basic Algorithms, Assignment 3, Solutions

Due, Thursday, Sept 27

1. Page 67 Exercise 1 (Think of n large. Possible answers are: Negligible effect, takes (roughly) twice as long, takes much much longer.)
   Solution:
   n^2: Double, times 4; Add one, negligible
   n^3: Double, times 8; Add one, negligible
   $100n^2$: Double, times 4; Add one, negligible (same as n^2)
   $n \lg n$: Double, times 2; Add one, negligible (when you double n the $\lg n$ only goes up by 1 which is a negligible effect.)
   2^n: Double, much much longer (e.g., million to trillion), add 1, times 2.

2. Page 67 Exercise 2

3. Page 67 Exercise 3
   Solution: $\sqrt{10n}$, $n + 10$, $n^2 \ln n$, $n^{2.5}$, 10^n, 100^n. Ignoring constant terms the first four are $n^{1/2}, n, n^2 \ln n, n^{2.5}$. The bigger power of n means the bigger function, the logs don't count unless there is a tie (for example, $n = o(n \ln n)$.)

4. Page 68 Exercise 6
   Solution:
   (a) This is $O(n^3)$ as there are $\leq n^2$ choices of i, j and for each one you are adding at most n numbers.
   (b) This is also $\Omega(n^3)$. One way to see this is to look at all the choices of i with $1 \leq i \leq n/3$ and j with $2n/3 \leq j \leq n$. This is $n^2/9$ choices and for each one you have to add at least $n/3$ terms (all the middle ones) so the total time is at least $n^3/27$. This is an argument we will see again. It doesn't actually give the right constant but in ThetaWorld we don't care.
   (c) One way (there are others) is for each i to keep a cumulative count for the $A[i, j]$ as j increases. That way, for each j you only need one more addition. As it is taking constant time for each entry this is a $O(n^2)$ algorithm, *much better* than $\Theta(n^3)$. Here is an implementation:
   For $i = 1, 2, \ldots, n-1$
       $\texttt{SUM} \leftarrow A[i]$
       For $j = i + 1, i + 2, \ldots, n$
           $\texttt{SUM} \leftarrow \texttt{SUM} + A[j]$
           $A[i, j] \leftarrow \texttt{SUM}$

Endfor
Endfor

5. Let $A[i]$, $0 \le i \le n$, represent an $n+1$-digit decimal number α Give an algorithm (repeating what you learned in fifth grade) for dividing α by 7 (just for definiteness). Your output should be the quotient as an array $Q[i]$ and a remainder R. How long does this algorithm take in ThetaWorld. (e.g.: $\Theta(n), \Theta(n \lg n), \Theta(n^2)$ or what?)
   Solution:
   $R \leftarrow 0$
   For $i = n, n-1, \dots 0$
   $X \leftarrow A[i] + 10 * R$ (*add the carry*)
   $Q[i] \leftarrow \lfloor X/7 \rfloor$ (*quotient*)
   $R \leftarrow X - 7 * Q[i]$ (*new remainder*)
   Endfor (* final R will be desired remainder *)

   X will be at most 9 (a single digit) plus 6 the biggest remainer so finding $Q[i], R$ take constant time. There is only a single run through the digits so this is a linear time, that is, $\Theta(n)$ time algorithm.

6. Let n be a power of two, $n = 2^t$. Let p be a prime number. Our object is to calculate $7^n \bmod p$. We think of p as being of moderate size but n being very large. (For example, $p = 137$, $n = 2^{30}$.) *Assume* that we have a subroutine that multiplies two numbers modulo p in constant time.

   (a) `Method I:` Start with 1 and multiply by 7 until reaching 7^n. How long does this take. (Answer in ThetaWorld as a function of n.)
   Solution: There are n multiplications (well, actually $n-1$ but these are asymptotically the same) and each takes (by assumption!) constant time giving a $\Theta(n)$ algorithm.

   (b) `Method II:` Start with 7 and square until reaching 7^n. How long does this take. (Answer in ThetaWorld as a function of n.)
   Solution: This *islandhopping* strategy goes 7, 7^2, 7^4, 7^8,... and reaches 7^n in t multiplications so the time is $\Theta(t)$. But as $n = 2^t$ we have $t = \lg n$ so in terms of n this is a $\Theta(\lg n)$ algorithm. Method II is *far superior!*

If you take a number and double it and double it again and then double it a few more times, the number gets bigger and bigger and goes higher and higher and only arithmetic can tell you what

the number is when you quit doubling.
from *Arithmetic* by Carl Sandburg