Graph Theory:

a) Combinatorial Structure

b) Algebraic Structure ~ Spectral Properties

c) Probabilistic Structure ~
Random Graphs & Their Evolution.

Social Interactions (Pair-wise)

- Graph Theory (Interaction Choices)
- Game Theory (Strategic Choices)

We wish to model Strategic Interactions among Rational Agents.

Ingredients:

$V \equiv$ Set of Actors

$E \subseteq V \times V \equiv$ Set of Links

$S_v \equiv$ Strategy space $v \in V$

$u_v : \prod_{v \in V} S_v \to \mathbb{R}_+ \equiv$ Pay-off functions of $v \in V$

$(V, E, S_v|_{v \in V}, u_v|_{v \in V})$
determine

A Social Network and how its agents behave.

Defn GRAPHS (Networks)

A graph $G = (V, E)$ consists of a set of vertices V together with a set of edges $E \subseteq V \times V$.

$\Rightarrow$ A mathematical object describing an irreflexive, symmetric binary relation on a discrete set, which need not be finite.

Example: Friendship.

IRREFLEXIVE : One is not his own friend.

$\langle v, v \rangle \notin E$ (no self-loop)

SYMMETRIC : One is a friend to a friend.

$\langle v, w \rangle \in E \Leftrightarrow \langle w, v \rangle \in E$

NON TRANSITIVE : One is not necessarily a friend to a friend's friend.

$\langle u, v \rangle \in E \wedge \langle v, w \rangle \in E \not\Rightarrow \langle u, w \rangle \in E$.

Friendship relation in a Social Network can be described by an undirected graph.

- An edge $e = (u, v) \in E$ (where $E \subseteq V \times V$) is described by the unordered pair of vertices (players), which serve as edge's end-points.

◇ Two vertices ~~are~~ u and v are adjacent if $\exists_e \ e=(u,v)$ connecting u and v.

◇ Notation: $|V| = n = \#\text{vertices}$
$|E| = m = \#\text{edges}$.

$$m \le \binom{n}{2} = \frac{n(n-1)}{2} \quad \begin{cases} n = \#\text{ ways to choose } u \\ n-1 = \#\text{ ways to choose } v \\ \text{identifying edge} \\ (u,v) \equiv (v,u) \text{ [Symm]} \end{cases}$$

STRICT GRAPH:

No self-loop $(u,u) \notin E \quad \forall_u$

No multi-edge $e_1 = (u_1, v_1) \neq e_2(u_2, v_2)$

$$\Rightarrow (u_1 = u_2) \Rightarrow (v_1 \neq v_2)$$
$$\wedge (u_1 = v_2) \Rightarrow (v_1 \neq u_2)$$
$$\forall e_1, e_2$$

Two vertices u, and v are adjacent

$$\exists_e \ e = (u,v)$$

Two edges e and f are incident

$$\exists_u \ e = (u,v) \wedge f = (u,w)$$

$$d(v) = |\{u \mid (u,v) \in E\}| = \text{Degree.}$$

The number of vertices adjacent to a given vertex v is called the degree of the vertex.

$$\sum_{v \in V} d(v) = 2|E| = 2m$$

Average degree of the graph

$$\bar{d} = \frac{\sum_{v \in V} d(v)}{V} = \frac{2m}{n}$$

Density = Ratio of the number of edges to the number of possible total

$$= \frac{|E|}{\binom{n}{2}} = \frac{2m}{n(n-1)} = \frac{\Sigma d(v)}{n(n-1)} = \frac{\bar{d}}{n-1}$$

Density = 1 $\Rightarrow$ Graph is $\boxed{\text{complete.}}$

$$\bar{d} = (n-1) \qquad \forall v \; d(v) \le n-1$$

$$\Rightarrow \forall_v \; d(v) = n-1$$

A graph is complete if all of its vertices are adjacent to all others.

- If a social network has many "well-connected individuals" then the network is "dense," since

$$\text{density} = \bar{d}/n-1$$

$\Rightarrow$ Finding and connecting to "well-connected" subnetworks is beneficial.

CLIQUE CLUB

CLAN.

Distance between two individuals:

Path: A sequence of adjacent distinct vertices

$$v_0, v_1, \ldots, v_n$$
$$\forall_i \quad (v_i, v_{i+1}) \in E, \quad 0 \leq i < n$$
$$\forall_{i,j} \quad v_i \neq v_j \qquad 0 \leq i \neq j \leq n$$

in which no vertex occurs more than once.

Path $\leq$ Trail $\leq$ Walk

Path: Vertices are distinct
Trail: Edges are distinct
Walk: Neither.

- The (geodesic)-distance between two vertices is the length of the shortest path connecting them.

$d(u,v)$ = Geodesic distance between u and v.

- The maximal geodesic-distance in a graph is its <u>diameter</u>, $\mathcal{D}(G)$

$$\forall_{u,v} \quad d(u,v) \leq \mathcal{D}(G).$$

$$G_1(V_1, E_1) \subseteq G(V,E) \quad (\text{Subgraph})$$
$$\text{iff} \quad V_1 \subseteq V \ \wedge \ E_1 \subseteq E.$$

- A subgraph of a graph G is a graph whose vertices and edges are contained in G.

$$G_1 \subseteq G, \quad |V_1| = k, \quad |E_1| = \binom{k}{2}$$

- A clique is a maximal complete subgraph.

- The subgraph $G(S)$ of a graph G induced by the set of nodes S is defined as the maximal subgraph of G that has vertex set S.