LECTURE #3

February 18 2014.

1) Basic Framework:

Sender Receiver Game.

2-player Information-Asymmetric Game between an Informed Agent and an Uninformed Agent.

$$\underset{\text{(Sender)}}{S} \qquad\qquad \underset{\text{(Receiver)}}{R}$$

$$D_S \longrightarrow M \longrightarrow A_S$$

$$u_S(D, M, A) \qquad\qquad u_R(D, M, A)$$

Sender has a map from state to message. Receiver: map from message to Action.

They have independent utility functions.

2) Deception: 2+v+r - games.

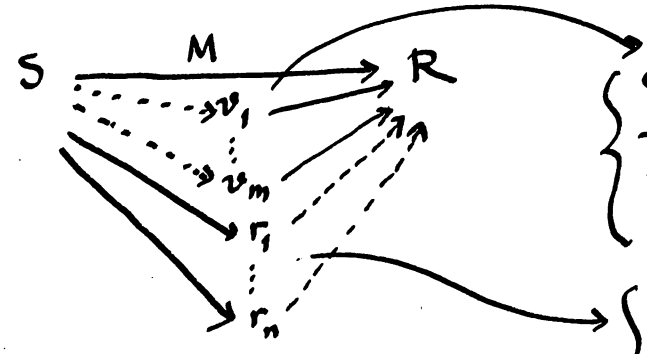
Introduce a set of verifiers, $v_1, v_2, \ldots, v_m$

a set of recommenders, $r_1, r_2, \ldots, r_n$

Map-Reduce

Online Algorithms "BALANCE"

Machine Learning



crowd-sourcing
fact-checkers
regulators
reviewers

advertisers
recommenders
market makers

There are cost functions associated with the verifiers + recommenders, etc.

3) The sizes of $|v|$ & $|r|$ can be controlled. $|v| = |r| = 0$

$\Rightarrow$ 1-click or "I am feeling lucky"

Note: The edges in social network are not necessary of communication – only for establising trust.

Simplest Example: GOOGLE.
"Search"

$S \longrightarrow M$: Search word $\longrightarrow R$

$\mathcal{D}_S$ = Data

$\mathcal{U} \setminus \mathcal{D}_S$ = State of ignorance

R: Select a page from $\mathcal{U} \setminus \mathcal{D}_S$ containing M, etc. {+ Recommended Hyperlinks} $\mathcal{L}_1$

$S \longrightarrow M \in \mathcal{L}_1 \longrightarrow R$

R: Page corresponding to M {+ Recomended Hyperlinks} $\mathcal{L}_2$

⋮

etc.

until $\mathcal{L}_k = \phi$ or $\mathcal{L}_k$ is no longer utility-improving.

→ Start with a new search word.

⇒ Random Surfer with Teleportation.

METRICS FOR A SOCIAL NETWORK.

$G = (V, E)$: V = Set of Vertices
$E \subseteq V \times V$ = Set of Edges.

Metrics.

1) Size, $n = |V|$
2) Density, $\frac{m}{\binom{n}{2}}$ = Density $= \frac{n\bar{d}/2}{n(n-1)/2} = \frac{\bar{d}}{n-1}$

   ($m = |E|$: number of edges,
   $\bar{d}$ = Average local degree, ...)
3) Spectral Gap, $1-\lambda_2(G)$
   $\lambda_2(\mathcal{G})$ = Second Eigenvalue of the Google Matrix $\mathcal{G}$ of the graph.

   {Also, related to the eigenvalues of the laplacian of the graph $G = (V, E)$ }

4) Graph Width, Expansion Factor, Cheeger-Constants, etc. Degrees of Separation.

Focus on : 2) density, $\bar{d}$ $\rightarrow$ Random Graph Models.
3) spectral properties, $\lambda_1(\mathcal{L}(G))$. $\rightarrow$ Random Surfer Models.

[Note: a) We are assuming that sender and receiver will keep their utility and data private.]
b) Utility may not be known to the users.
BPC models, Prospect Theory, etc.
[Belief Preference & Constraints]
c) Data may not be fully known to the users
d) Users may not be fully rational.

$G = (V, E)$, $E \subseteq V \times V$
= Irreflexive, Symmetric, Nontransitive Binary Relation.

A = Adjacency Matrix of $G \in \{0,1\}^{n \times n}$

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$A^T = A$.

D = Diagonal Degree Matrix of G

$$d_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ 0 & \text{otherwise}. \end{cases}$$

Boundary Matrix B.

* First turn the edges into directed edges by choosing the directions arbitrarily.

B has its columns idexed by the vertices of G and rows, indexed by the edges of G.

$$B(e, v) = \begin{cases} 1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e \\ 0 & \text{otherwise}. \end{cases}$$

$$\boxed{B^T B = L = D - A}$$

$$\Delta = (I - P) = D^{-1}(D - A)$$

$$P(u, v) = \begin{cases} 1/d_v & \text{if } (u, v) \in E \\ 0 & \text{otherwise}. \end{cases}$$

$$\mathcal{L} = D^{1/2} \Delta D^{-1/2} = \text{Laplacian}.$$

Normalized Boundary Matrix $\tilde{B}$

$$\tilde{B}(e,v) = \begin{cases} 1/\sqrt{d_v} & \text{if } v = \text{head of } e \\ -1/\sqrt{d_v} & \text{if } v = \text{tail of } e \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{L} = \tilde{B}^T \tilde{B}$$

$$\left.\begin{array}{l}\text{Graph}\\ \text{Topology}\end{array}\right\} \qquad \begin{array}{c} C_1 \underset{\tilde{B}^T}{\overset{\tilde{B}}{\rightleftarrows}} C_0 \end{array}$$

C_0 → 0-chains (vertices V)

C_1 → 1-chains (edges, E).

$$g : V \to \mathbb{R}$$

$$\mathcal{L}\, g(u) = D^{1/2} \Delta D^{-1/2} g(u)$$

$$= \frac{1}{\sqrt{d_u}} \sum_{\substack{v \\ u \sim v}} \left(\frac{g(u)}{\sqrt{d_u}} - \frac{g(v)}{\sqrt{d_v}} \right)$$

$$f = D^{-1/2} g : V \to \mathbb{R}$$

Rayleigh Quotient:
$$\frac{\langle g, \mathcal{L} g\rangle}{\langle g, g\rangle} = \frac{\langle g, D^{-1/2} L D^{1/2} g\rangle}{\langle g, g\rangle}$$

$$= \frac{\langle f, \mathcal{L} f\rangle}{\langle D^{1/2} f, D^{1/2} f\rangle}$$

$$= \frac{\langle f, B^T B f\rangle}{\langle D^{1/2} f, D^{1/2} f\rangle}$$

$$= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v}$$

Let ϕ_i; $i = 0, 1, \ldots, n-1$ denote the orthonormal eigen functions of $\mathcal{L}$

$$\mathcal{L} = \sum_{i=0}^{n-1} \lambda_i \phi_i^T \phi_i = \Phi^T \Lambda \Phi$$

$$\langle g, \mathcal{L} g \rangle = \langle g, \Phi^T \Lambda \Phi g \rangle = \langle \Phi g, \Lambda \Phi g \rangle$$

$$= \langle h, \Lambda h \rangle = \sum_{j=0}^{n-1} \lambda_j |h_j|^2$$

$$R(g) = \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} = \frac{\langle \Phi g, \Lambda \Phi g \rangle}{\langle g, g \rangle} = \sum_{i=0}^{n-1} \lambda_i |h_i|^2 \Big/ \sum_{i=1}^{n-1} |h_i|^2$$

$$= R_\Phi(h)$$

Let the eigenvalues be ordered:

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$$

Maxi Min Principle:

$$\min_{g \neq 0} R(g) = \lambda_0$$

$$\max_{k_1 \neq 0} \min_{\langle g, k_1 \rangle = 0} R(g) = \lambda_1$$

$$\vdots$$

$$\max_{k_1, k_2, \ldots, k_p \neq 0} \; \min_{\substack{\langle g, k_1 \rangle = 0 \\ \langle g, k_2 \rangle = 0 \\ \vdots \\ \langle g, k_p \rangle = 0}} R(g) = \lambda_{p+1}$$

$\Phi_0 = D^{1/2} \mathbb{1}$ Eigenfunction

$$f = D^{-1/2} \Phi_0 = \mathbb{1}$$

$$\lambda_0 = \sum_{u \sim v} (f(u) - f(v))^2 \Big/ \sum_v f(v)^2 d_v = 0$$

$$\lambda_1 = \min_{f \perp \Phi_0} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v}$$

$$= \min_f \max_t \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v (f(v) - t)^2 d_v}$$

$$= \min_f \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v (f(v) - \langle f \rangle)^2 d_v}, \quad \text{where } \langle f \rangle = \frac{\sum_u f(u) d_u}{\underbrace{\sum_v d_v}_{\text{vol } G}}$$

$$\lambda_1 = \text{vol } G \min_f \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_{u,v} (f(u) - f(v))^2 d_u d_v}$$

$G = (V, E)$; $S \subset V$ and $\bar{S} = V \setminus S$

∂S = The edge boundary of S consists of all edges with exactly one end point in S

$$\partial S = \{ (u,v) \in E \mid u \in S \wedge v \in \bar{S} \}$$

$$\text{vol } S = \sum_{u \in S} d_u$$

$$\text{vol } G = \sum d_u = 2m$$

Cheeger Ratio $\quad c(S) = \dfrac{|\partial S|}{\min\{\text{vol}(S), \text{vol}(G) - \text{vol}(S)\}}$

Cheeger Constant $\quad c_G = \min_S c(S)$

Cheeger Inequality

$$2 c_G \geq \lambda_1 \geq \frac{c_G^2}{2}$$

$$\lambda_1 = \min_{f \perp \Phi_0} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v}$$

Take $\hat{f}(u) = \begin{cases} 1/\text{vol } S & \text{if } u \in S \\ -1/\text{vol } \bar{S} & \text{if } u \in \bar{S} \end{cases}$

$$\lambda_1 \leq \frac{|\partial S| \left\{ \frac{1}{\text{vol} S} + \frac{1}{\text{vol} \bar{S}} \right\}^2}{\text{vol} S \left(\frac{1}{\text{vol} S^2} \right) + \text{vol} \bar{S} \left(\frac{1}{\text{vol} \bar{S}^2} \right)}$$

$$\leq |\partial S| \left\{ \frac{1}{\text{vol} S} + \frac{1}{\text{vol} \bar{S}} \right\}$$

$$\leq \frac{2 |\partial S|}{\min(\text{vol} S, \text{vol} \bar{S})} = 2 C_G .$$

PERSONALIZED PAGE RANK:

Two parameters $\begin{cases} \text{a seed vector, } S \\ \text{a jumping constant, } \alpha. \end{cases}$

$$pr(\alpha, s) = \alpha s + (1-\alpha)\, pr(\alpha, s) W$$

$W = \frac{I + P}{2}$ = Denotes a lazy walk of G (Exploit vs Explore).

s: An initial probability distribution

α = A positive value - to scale the rate of propagation.

$$p = pr(\alpha, s) = \alpha \sum_{k=0}^{\infty} (1-\alpha)^k s W^k$$
$$= \alpha s \sum_{k=0}^{\infty} (1-\alpha)^k W^k$$

$$p\left[I - (1-\alpha) W\right] = \alpha s \left[I - (1-\alpha) W\right] \sum_{k=0}^{\infty} (1-\alpha)^k W^k$$
$$= \alpha s \left[I - (1-\alpha) W + (1-\alpha) W - (1-\alpha)^2 W^2 \dots\right]$$
$$= \alpha s$$

$$p\left[I - (1-\alpha) \frac{I+P}{2}\right] = \alpha s$$

$$p\left[2I - I - P + \alpha (I+P)\right] = 2\alpha s$$

$$p\left[2\alpha I + (1-\alpha) I - (1-\alpha) P\right] = 2\alpha s$$

$$\beta = \frac{2\alpha}{1-\alpha} \qquad p\left[\beta I + (I-P)\right] = \beta s$$

$$p\left[\beta I + \Delta\right] = \beta s$$
$$p\left[\beta I + \mathcal{L}\right] = \beta s$$

$$\mathcal{L} = D^{1/2} \Delta D^{-1/2}$$
$$= \sum_{i=1}^{n-1} \lambda_i \Phi_i^T \Phi_i$$

$$\beta I + \mathcal{L}$$
$$= \sum_{i=1}^{n-1} (\beta + \lambda_i) \Phi_i^T \Phi_i$$

$$G_\beta = \sum_{i=1}^{n-1} \frac{1}{\beta + \lambda_i} \Phi_i^T \Phi_i$$

= Discrete Green's Formula.

$$pr(\alpha, s) = \beta s G_\beta$$

G_β can be computed iteratively (using MAP-REDUCE)

$$(1+\beta) G_\beta = I + G_\beta P$$