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# THREE-LEVEL BDDC IN THREE DIMENSIONS

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**Abstract.** BDDC methods are nonoverlapping iterative substructuring domain decomposition methods for the solution of large sparse linear algebraic systems arising from discretization of elliptic boundary value problems. Its coarse problem is given by a small number of continuity constraints which are enforced across the interface. The coarse problem matrix is generated and factored by direct solvers at the beginning of the computation and it can ultimately become a bottleneck, if the number of subdomains is very large. In this paper, two three-level BDDC methods are introduced for solving the coarse problem approximately in three dimensions. This is an extension of previous work for the two dimensional case and since vertex constraints alone do not suffice to obtain polylogarithmic condition number bound, edge constraints are considered in this paper. Some new technical tools are then needed in the analysis and this makes the three dimensional case more complicated than the two dimensional case. Estimates of the condition numbers are provided for two three-level BDDC methods and numerical experiments are also discussed.

**Key words.** BDDC, three-level, three dimensions, domain decomposition, coarse problem, condition number, Chebyshev iteration

**AMS subject classifications.** 65N30, 65N55

**1. Introduction.** BDDC (Balancing Domain Decomposition by Constraints) methods, which were introduced and analyzed in [3, 7, 8], are similar to the balancing Neumann-Neumann algorithms. However, the coarse problem, in a BDDC algorithm, is given in terms of a set of primal constraints and the matrix of the coarse problem is generated and factored by using direct solvers at the beginning of the computation. We note that there are now computer systems with more than 100,000 powerful processors, which allow very large and detailed simulations. The coarse component of a two-level preconditioner can therefore be a bottleneck if the number of subdomains is very large. One way to remove this difficulty is to introduce one or more additional levels. In our recent paper [11], two three-level BDDC methods were introduced for two dimensional problems with vertex constraints. We solve the coarse problem approximately, by using the BDDC idea recursively, while a good rate of convergence still can be maintained. However, in three dimensional space, vertex constraints alone are not enough to obtain good polylogarithmic condition number bound due to much weaker interpolation estimate and constraints on the averages over edges or faces are needed. The new constraints lead to a considerably more complicated coarse problem and the need for new technical tools in the analysis. In this paper, we extend the two three-level BDDC methods in [11] to the three dimensional case using primal edge average constraints. With the help of the new technical tools, we provide estimates of the condition number bounds of the system with these two new preconditioners.

The rest of the paper is organized as follows. We first review the two-level BDDC methods briefly in Section 2. We introduce our first three-level BDDC method and the corresponding preconditioner $\widetilde{M}^{-1}$ in Section 3. We give some auxiliary results in Section 4. In Section 5, we provide an estimate of the condition number bound for the system with the preconditioner $\widetilde{M}^{-1}$ which is of the form $C\left(1+\log\frac{\hat{H}}{H}\right)^2\left(1+\log\frac{H}{h}\right)^2$, where $\hat{H}$, $H$, and $h$ are typical diameters of the subregions, subdomains, and elements,

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respectively. In Section 6, we introduce a second three-level BDDC method which uses Chebyshev iterations. We denote the corresponding preconditioner by $\widehat{M}^{-1}$. We show that the condition number bound of the system with the preconditioner $\widehat{M}^{-1}$ is of the form $CC(k)\left(1+\log \frac{H}{h}\right)^2$, where $C(k)$ is a function of $k$, the number of Chebyshev iterations, and also depends on the eigenvalues of the preconditioned coarse problem, and the two parameters chosen for the Chebyshev iteration. $C(k)$ goes to 1 as $k$ goes to $\infty$, i.e., the condition number approaches that of the two-level case. Finally, some computational results are presented in Section 7.

**2. The two-level BDDC method.** We will consider a second order scalar elliptic problem in a three dimensional region $\Omega$: find $u \in H_0^1(\Omega)$, such that

$$\int_\Omega \rho \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where $\rho(x) > 0$ for all $x \in \Omega$. We decompose $\Omega$ into $N$ nonoverlapping subdomains $\Omega_i$ with diameters $H_i$, $i = 1, \cdots, N$, and set $H = \max_i H_i$. We then introduce a triangulation for all the subdomains. Let $\Gamma$ be the interface between the subdomains and the set of interface nodes $\Gamma_h$ is defined by $\Gamma_h = (\cup_{i \neq j} \partial\Omega_{i,h} \cap \partial\Omega_{j,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$. The nodes of the different triangulations match across $\Gamma$.

Let $\mathbf{W}^{(i)}$ be the standard finite element space of continuous, piecewise trilinear functions on $\Omega_i$; the algorithms and theories developed in this paper work for other lower order finite elements as well. We assume that these functions vanish on $\partial\Omega$. Each $\mathbf{W}^{(i)}$ can be decomposed into a subdomain interior part $\mathbf{W}_I^{(i)}$ and a subdomain interface part $\mathbf{W}_\Gamma^{(i)}$, i.e., $\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma^{(i)}$, where the subdomain interface part $\mathbf{W}_\Gamma^{(i)}$ will be further decomposed into a primal subspace $\mathbf{W}_\Pi^{(i)}$ and a dual subspace $\mathbf{W}_\Delta^{(i)}$, i.e., $\mathbf{W}_\Gamma^{(i)} = \mathbf{W}_\Pi^{(i)} \bigoplus \mathbf{W}_\Delta^{(i)}$.

We denote the associated product spaces by $\mathbf{W} := \prod_{i=1}^N \mathbf{W}^{(i)}$, $\mathbf{W}_\Gamma := \prod_{i=1}^N \mathbf{W}_\Gamma^{(i)}$, $\mathbf{W}_\Delta := \prod_{i=1}^N \mathbf{W}_\Delta^{(i)}$, $\mathbf{W}_\Pi := \prod_{i=1}^N \mathbf{W}_\Pi^{(i)}$, and $\mathbf{W}_I := \prod_{i=1}^N \mathbf{W}_I^{(i)}$. Correspondingly, we have $\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma$, and $\mathbf{W}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta$.

We will consider elements of $\mathbf{W}$ which are discontinuous across the interface. However, the finite element approximation of the elliptic problem is continuous across $\Gamma$. We denote the corresponding subspace of $\mathbf{W}$ by $\widehat{\mathbf{W}}$.

We further introduce an interface subspace $\widetilde{\mathbf{W}}_\Gamma \subset \mathbf{W}_\Gamma$, for which certain primal constraints are enforced. Here, we only consider the case of edge average constraints over all the edges of all subdomains. We change the variables to make the edge average degrees of freedom explicit, see [4, Sec 6.2] and [5, Sec 2.3]. From now on, we assume all the matrices are written in terms of the new variables. The continuous primal subspace denoted by $\widehat{\mathbf{W}}_\Pi$ is spanned by the continuous edge average variables of each edge of the interface. The space $\widetilde{\mathbf{W}}_\Gamma$ can be decomposed into $\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta$.

The global problem has the form: find $(\mathbf{u}_I, \mathbf{u}_\Delta, \mathbf{u}_\Pi) \in (\mathbf{W}_I, \widehat{\mathbf{W}}_\Delta, \widehat{\mathbf{W}}_\Pi)$, such that

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & A_{\Pi I}^T \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Pi\Delta}^T \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{pmatrix}.$$

This problem is assembled from the subdomain problems

$$\begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^{(i)} \\ \mathbf{u}_\Delta^{(i)} \\ \mathbf{u}_\Pi^{(i)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I^{(i)} \\ \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{pmatrix}.$$

We also denote by $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$, the right hand side spaces corresponding to $\mathbf{W}_\Gamma$, $\widehat{\mathbf{W}}_\Gamma$, and $\widetilde{\mathbf{W}}_\Gamma$, respectively.

In order to describe BDDC algorithm, we need to introduce several restriction, extension, and scaling operators between different spaces. The restriction operator $R_\Gamma^{(i)}$ maps a vector of the space $\widehat{\mathbf{W}}_\Gamma$ to a vector of the subdomain subspace $\mathbf{W}_\Gamma^{(i)}$. Each column of $R_\Gamma^{(i)}$ with a nonzero entry corresponds to an interface node, $x \in \partial\Omega_{i,h} \cap \Gamma_h$, shared by the subdomain $\Omega_i$ and its next neighbor subdomains. $\overline{R}_\Gamma^{(i)}$ is similar to $R_\Gamma^{(i)}$, and represents the restriction from $\widetilde{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Gamma^{(i)}$. $R_\Delta^{(i)} : \mathbf{W}_\Delta \to \mathbf{W}_\Delta^{(i)}$, is the restriction matrix which extracts the subdomain part, in the space $\mathbf{W}_\Delta^{(i)}$, of the functions in the space $\mathbf{W}_\Delta$. $R_\Pi^{(i)}$ is the restriction operator from the space $\widehat{\mathbf{W}}_\Pi$ to $\mathbf{W}_\Pi^{(i)}$. Multiplying each such element of $R_\Gamma^{(i)}$, $\overline{R}_\Gamma^{(i)}$, and $R_\Delta^{(i)}$ with $\delta_i^\dagger(x)$ gives us $R_{D,\Gamma}^{(i)}$, $\overline{R}_{D,\Gamma}^{(i)}$, and $R_{D,\Delta}^{(i)}$, respectively. Here, we define $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$, $\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j\in\mathcal{N}_x} \rho_j^\gamma(x)}$, $x \in \partial\Omega_{i,h} \cap \Gamma_h$, where $\mathcal{N}_x$ is the set of indices $j$ of the subdomains such that $x \in \partial\Omega_j$ and $\rho_j(x)$ is the coefficient of (2.1) at $x$ in the subdomain $\Omega_j$. Furthermore, $R_{\Gamma\Delta}$ and $R_{\Gamma\Pi}$ are the restriction operators from the space $\widetilde{\mathbf{W}}_\Gamma$ onto its subspace $\mathbf{W}_\Delta$ and $\mathbf{W}_\Pi$ respectively. $R_\Gamma : \widehat{\mathbf{W}}_\Gamma \to \mathbf{W}_\Gamma$ and $\overline{R}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \mathbf{W}_\Gamma$ are the direct sums of $R_\Gamma^{(i)}$ and $\overline{R}_\Gamma^{(i)}$, respectively. $\widetilde{R}_\Gamma : \widehat{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{W}}_\Gamma$ is the direct sum of $R_{\Gamma\Pi}$ and the $R_\Delta^{(i)} R_{\Gamma\Delta}$. The scaled operators $R_{D,\Gamma}$ and $R_{D,\Delta}$ are the direct sums of $R_{D,\Gamma}^{(i)}$ and $R_{D,\Delta}^{(i)}$, respectively. $\widetilde{R}_{D,\Gamma}$ is the direct sum of $R_{\Gamma\Pi}$ and $R_{D,\Delta} R_{\Gamma\Delta}$.

We also use the same restriction, extension, and scaled restriction operators for the right hand side spaces $\mathbf{F}_\Gamma$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_\Gamma$.

We define an operator $\widetilde{S}_\Gamma : \widetilde{\mathbf{W}}_\Gamma \to \widetilde{\mathbf{F}}_\Gamma$, which is of the form: given $\mathbf{u}_\Gamma = \mathbf{u}_\Pi \oplus \mathbf{u}_\Delta \in \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widetilde{\mathbf{W}}_\Gamma$, find $\widetilde{S}_\Gamma \mathbf{u}_\Gamma \in \widetilde{\mathbf{F}}_\Gamma$ by eliminating the interior variables of the system:

$$\begin{aligned} & A\left(\mathbf{u}_I^{(1)}, \mathbf{u}_\Delta^{(1)}, \cdots, \mathbf{u}_I^{(N)}, \mathbf{u}_\Delta^{(N)}, \mathbf{u}_\Pi\right)^T \\ & = \left(\mathbf{0}, R_\Delta^{(1)} R_{\Gamma\Delta} \widetilde{S}_\Gamma \mathbf{u}_\Gamma, \cdots, \mathbf{0}, R_\Delta^{(N)} R_{\Gamma\Delta} \widetilde{S}_\Gamma \mathbf{u}_\Gamma, R_{\Gamma\Pi} \widetilde{S}_\Gamma \mathbf{u}_\Gamma\right)^T, \end{aligned} \tag{2.2}$$

where $A$ is of the form

$$\begin{pmatrix} A_{II}^{(1)} & A_{\Delta I}^{(1)^T} & \cdots & \cdots & \cdots & A_{\Pi I}^{(1)^T} R_\Pi^{(1)} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & \cdots & \cdots & \cdots & A_{\Pi\Delta}^{(1)^T} R_\Pi^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & A_{II}^{(N)} & A_{\Delta I}^{(N)^T} & A_{\Pi I}^{(N)^T} R_\Pi^{(N)} \\ \cdots & \cdots & \cdots & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & A_{\Pi\Delta}^{(i)^T} R_\Pi^{(i)} \\ R_\Pi^{(1)^T} A_{\Pi I}^{(1)} & R_\Pi^{(1)^T} A_{\Pi\Delta}^{(1)} & \cdots & R_\Pi^{(N)^T} A_{\Pi I}^{(N)} & R_\Pi^{(N)^T} A_{\Pi\Delta}^{(N)} & \sum_{i=1}^N R_\Pi^{(i)^T} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)} \end{pmatrix}.$$

The reduced interface problem can be written as: find $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$ such that

$$\widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \mathbf{u}_\Gamma = \mathbf{g}_\Gamma,$$

where the operator $\widetilde{S}_\Gamma$ is defined in Equations (2.2), and

$$\mathbf{g}_\Gamma = \sum_{i=1}^N R_\Gamma^{(i)^T} \left\{ \begin{pmatrix} \mathbf{f}_\Delta^{(i)} \\ \mathbf{f}_\Pi^{(i)} \end{pmatrix} - \begin{pmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{pmatrix} A_{II}^{(i)^{-1}} \mathbf{f}_I^{(i)} \right\}.$$

The two-level BDDC method is of the form

$$M^{-1} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \mathbf{u}_\Gamma = M^{-1} \mathbf{g}_\Gamma,$$

where the preconditioner $M^{-1} = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma}$ has the following form:

(2.3)
$$\widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \left( \sum_{i=1}^N \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} \right) R_{\Gamma\Delta} + \Phi S_\Pi^{-1} \Phi^T \right\} \widetilde{R}_{D,\Gamma}.$$

Here $\Phi$ is the matrix given by the coarse level basis functions with minimal energy, and it is defined by

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{i=1}^N \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} R_\Pi^{(i)}.$$

The coarse level problem matrix $S_\Pi$ is determined by

(2.4)
$$S_\Pi = \sum_{i=1}^N R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)},$$

which is obtained by assembling subdomain matrices; for additional details, cf. [3, 7, 5].

We know that, under certain assumptions, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma \le \mathbf{u}_\Gamma^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \mathbf{u}_\Gamma \le C \left(1 + \log(H/h)\right)^2 \mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma. \tag{2.5}$$

This can be established directly by using methods very similar to those of certain studies of the FETI-DP algorithms. Denote by $E_D$ and $P_D$, the average and jump operators (see [10, Formula (6.4) and (6.38)]), on the space $\widetilde{\mathbf{W}}_\Gamma$, respectively. Central to obtaining the condition number estimate for the preconditioned two-level BDDC operator is a bound for the $E_D$ operators (see [8, Theorem 25]). Since $E_D + P_D = I$ (see [10, Lemma 6.10]), we only need to find a bound for the $P_D$ operator. [10, Lemma 6.36] gives a bound for the $P_D$ operator under the assumptions [10, Assumption 4.3.1] for the triangulation and [10, Assumption 6.27.2] for the coefficient $\rho(x)$ of (2.1).

**3. A three-level BDDC method.** For the three-level cases, as in [11], we will not factor the coarse problem matrix $S_\Pi$ defined in (2.4) by a direct solver. Instead, we will solve the coarse problem approximately by using ideas similar to those for the two-level preconditioners.

We decompose $\Omega$ into $N$ subregions $\Omega^j$ with diameters $\hat{H}^j$, $j = 1, \cdots, N$. Each subregion $\Omega^j$ is the union of $N_j$ subdomains $\Omega_i^j$ with diameters $H_i^j$. Let $\hat{H} = \max_j \hat{H}^j$

and $H = \max_{i,j} H_i^j$, for $j = 1, \cdots, N$, and $i = 1, \cdots, N_j$. We introduce the subregional Schur complement

$$S_\Pi^{(j)} = \sum_{i=1}^{N_j} R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)},$$

and note that the coarse problem matrix $S_\Pi$ can be assembled from the $S_\Pi^{(j)}$.

Let $\widehat{\Gamma}$ be the interface between the subregions; $\widehat{\Gamma} \subset \Gamma$. We denote the vector space corresponding to the subdomain edge average variables in $\Omega^i$, by $\mathbf{W}_c^{(i)}$. Each $\mathbf{W}_c^{(i)}$ can be decomposed into a subregion interior part $\mathbf{W}_{c,\widehat{I}}^{(i)}$ and a subregion interface part $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$, i.e., $\mathbf{W}_c^{(i)} = \mathbf{W}_{c,\widehat{I}}^{(i)} \bigoplus \mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$ , where the subregion interface part $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$ can be further decomposed into a primal subspace $\mathbf{W}_{c,\widehat{\Pi}}^{(i)}$ and a dual subspace $\mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, i.e., $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)} = \mathbf{W}_{c,\widehat{\Pi}}^{(i)} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}^{(i)}$ . We denote the associated product spaces by $\mathbf{W}_c := \prod_{i=1}^N \mathbf{W}_c^{(i)}$, $\mathbf{W}_{c,\widehat{\Gamma}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$, $\mathbf{W}_{c,\widehat{\Delta}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, $\mathbf{W}_{c,\widehat{\Pi}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{\Pi}}^{(i)}$, and $\mathbf{W}_{c,\widehat{I}} := \prod_{i=1}^N \mathbf{W}_{c,\widehat{I}}^{(i)}$ . Correspondingly, we have $\mathbf{W}_c = \mathbf{W}_{c,\widehat{I}} \bigoplus \mathbf{W}_{c,\widehat{\Gamma}}$ , and $\mathbf{W}_{c,\widehat{\Gamma}} = \mathbf{W}_{c,\widehat{\Pi}} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}$ . We denote by $\widehat{\mathbf{W}}_c$ the subspace of $\mathbf{W}_c$ of functions that are continuous across $\widehat{\Gamma}$.

We next introduce an interface subspace $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \subset \mathbf{W}_{c,\widehat{\Gamma}}$, for which primal constraints are enforced. Here, we only consider edge average constraints. We need to change the variables again for all the local coarse matrices corresponding to the edge average constraints. The continuous primal subspace is denoted by $\widehat{\mathbf{W}}_{c,\widehat{\Pi}}$. The space $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ can be decomposed into $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} = \widehat{\mathbf{W}}_{c,\widehat{\Pi}} \bigoplus \mathbf{W}_{c,\widehat{\Delta}}$ .

We also denote by $\mathbf{F}_{c,\widehat{\Gamma}}$, $\widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$, and $\widetilde{\mathbf{F}}_{c,\widehat{\Gamma}}$, the right hand side spaces corresponding to $\mathbf{W}_{c,\Gamma}$, $\widehat{\mathbf{W}}_{c,\Gamma}$, and $\widetilde{\mathbf{W}}_{c,\Gamma}$, respectively, and will use the same restriction, extension, and scaled restriction operators for $\mathbf{F}_{c,\widehat{\Gamma}}$, $\widehat{\mathbf{F}}_\Gamma$, and $\widetilde{\mathbf{F}}_{c,\widehat{\Gamma}}$.

We define our three-level preconditioner $\widetilde{M}^{-1}$ by

(3.1)

$$\widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \left( \sum_{i=1}^N \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} \right) R_{\Gamma\Delta} + \Phi \widetilde{S}_\Pi^{-1} \Phi^T \right\} \widetilde{R}_{D,\Gamma},$$

cf. (2.3), where $\widetilde{S}_\Pi^{-1}$ is an approximation of $S_\Pi^{-1}$ and is defined as follows: given $\mathbf{\Psi} \in \widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$, let $\mathbf{y} = S_\Pi^{-1}\mathbf{\Psi}$ and $\widetilde{\mathbf{y}} = \widetilde{S}_\Pi^{-1}\mathbf{\Psi}$. Here $\mathbf{\Psi} = \left(\mathbf{\Psi}_{\widehat{I}}^{(1)}, \cdots, \mathbf{\Psi}_{\widehat{I}}^{(N)}, \mathbf{\Psi}_{\widehat{\Gamma}}\right)^T$, $\mathbf{y} = \left(\mathbf{y}_{\widehat{I}}^{(1)}, \cdots, \mathbf{y}_{\widehat{I}}^{(N)}, \mathbf{y}_{\widehat{\Gamma}}\right)^T$, and $\widetilde{\mathbf{y}} = \left(\widetilde{\mathbf{y}}_{\widehat{I}}^{(1)}, \cdots, \widetilde{\mathbf{y}}_{\widehat{I}}^{(N)}, \widetilde{\mathbf{y}}_{\widehat{\Gamma}}\right)^T$.

To solve $S_\Pi \mathbf{y} = \mathbf{\Psi}$ by block factorization in the two-level case, we can write

(3.2)

$$\begin{pmatrix} S_{\Pi_{\widehat{I}\widehat{I}}}^{(1)} & \cdots & \cdots & S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(1)^T} \widehat{R}_{\widehat{\Gamma}}^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & S_{\Pi_{\widehat{I}\widehat{I}}}^{(N)} & S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(N)^T} \widehat{R}_{\widehat{\Gamma}}^{(N)} \\ \widehat{R}_{\widehat{\Gamma}}^{(1)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(1)} & \cdots & \widehat{R}_{\widehat{\Gamma}}^{(N)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(N)} & \sum_{i=1}^N \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \end{pmatrix} \begin{pmatrix} \mathbf{y}_{\widehat{I}}^{(1)} \\ \vdots \\ \mathbf{y}_{\widehat{I}}^{(N)} \\ \mathbf{y}_{\widehat{\Gamma}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Psi}_{\widehat{I}}^{(1)} \\ \vdots \\ \mathbf{\Psi}_{\widehat{I}}^{(N)} \\ \mathbf{\Psi}_{\widehat{\Gamma}} \end{pmatrix}.$$

We have

$$\mathbf{y}_{\widehat{I}}^{(i)} = S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \left( \mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \mathbf{y}_{\widehat{\Gamma}} \right), \tag{3.3}$$

and

$$\left( \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} (S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}) \widehat{R}_{\widehat{\Gamma}}^{(i)} \right) \mathbf{y}_{\widehat{\Gamma}} = \mathbf{\Psi}_{\widehat{\Gamma}} - \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)}.$$

In the three-level BDDC algorithm, we need to introduce several restriction, extension, and scaling operators between different subregion spaces. The restriction operator $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ maps a vector of the space $\widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$ to a vector of the subdomain subspace $\mathbf{W}_{c,\Gamma}^{(i)}$. Each column of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ with a nonzero entry corresponds to an interface node, $x \in \partial\Omega^i \cap \Omega^j$, shared by the subregion $\Omega_i$ and certain neighboring subregions. $\widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)}$ is similar to $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ which represents the restriction from $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ to $\mathbf{W}_{c,\widehat{\Gamma}}^{(i)}$. $\widehat{R}_{\widehat{\Delta}}^{(i)}$ is the restriction matrix which extracts the subregion part, in the space $\mathbf{W}_{c,\widehat{\Delta}}^{(i)}$, of the functions in the space $\mathbf{W}_{c,\widehat{\Delta}}$. Multiplying each such element of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$, $\widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)}$, and $\widehat{R}_{\widehat{\Delta}}^{(i)}$ with $\widehat{\delta}_i^{\dagger}(x)$ gives us $\widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(i)}$, $\widehat{\overline{R}}_{\widehat{D},\widehat{\Gamma}}^{(i)}$, and $\widehat{R}_{\widehat{D},\widehat{\Delta}}^{(i)}$, respectively. Here, we define $\widehat{\delta}_i^{\dagger}(x)$ as follows: for $\gamma \in [1/2, \infty)$, $\widehat{\delta}_i^{\dagger}(x) = \frac{\rho_i^{\gamma}(x)}{\sum_{j\in\mathcal{N}_x} \rho_j^{\gamma}(x)}$, $x \in \partial\Omega_H^i \cap \widehat{\Gamma}_H$, where $\mathcal{N}_x$ is the set of indices $j$ of the subdomains such that $x \in \partial\Omega_H^j$ and $\rho_j(x)$ is the coefficient of (2.1) at $x$ in the subregion $\Omega^j$. (In our theory, we assume the $\rho_i$ are constant in the subregions.) Furthermore, $\widehat{R}_{\widehat{\Gamma}\widehat{\Delta}}$ and $\widehat{R}_{\widehat{\Gamma}\widehat{\Pi}}$ are the restriction operators from the space $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ onto its subspace $\mathbf{W}_{c,\widehat{\Delta}}$ and $\mathbf{W}_{c,\widehat{\Pi}}$ respectively. $\widehat{R}_{\widehat{\Gamma}} : \widehat{\mathbf{W}}_{c,\widehat{\Gamma}} \to \mathbf{W}_{c,\widehat{\Gamma}}$ and $\widehat{\overline{R}}_{\widehat{\Gamma}} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \to \mathbf{W}_{c,\widehat{\Gamma}}$ are the direct sum of $\widehat{R}_{\widehat{\Gamma}}^{(i)}$ and $\widehat{\overline{R}}_{\widehat{\Gamma}}^{(i)}$, respectively. $\widehat{\widetilde{R}}_{\widehat{\Gamma}} : \widehat{\mathbf{W}}_{c,\Gamma} \to \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ is the direct sum of $\widehat{R}_{\widehat{\Gamma}\widehat{\Pi}}$ and the $\widehat{R}_{\widehat{\Delta}}^{(i)} \widehat{R}_{\widehat{\Gamma}\widehat{\Delta}}$. The scaled operators $\widehat{R}_{\widehat{D},\widehat{\Gamma}}$ and $\widehat{R}_{\widehat{D},\widehat{\Delta}}$ are the direct sums of $\widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(i)}$ and $\widehat{R}_{\widehat{D},\widehat{\Delta}}^{(i)}$. $\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}$ is the direct sum of $\widehat{R}_{\widehat{\Gamma}\widehat{\Pi}}$ and $\widehat{R}_{\widehat{D},\widehat{\Delta}} \widehat{R}_{\widehat{\Gamma}\widehat{\Delta}}$.

Let $T^{(i)} = S_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}}^{(i)} - S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)^T}$ and $T = diag(T^{(1)}, \cdots, T^{(N)})$. We then introduce a partial assembled Schur complement of $S_{\Pi}$, $\widetilde{T} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \to \widetilde{\mathbf{F}}_{c,\widehat{\Gamma}}$ by

$$\widetilde{T} = \widehat{\overline{R}}_{\widehat{\Gamma}}^{T} T \widehat{\overline{R}}_{\widehat{\Gamma}}, \tag{3.4}$$

and define $\mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$, by

$$\mathbf{h}_{\widehat{\Gamma}} = \mathbf{\Psi}_{\widehat{\Gamma}} - \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \mathbf{\Psi}^{(i)}. \tag{3.5}$$

The reduced subregion interface problem can be written as: find $\mathbf{y}_{\widehat{\Gamma}} \in \widehat{W}_{c,\widehat{\Gamma}}$, such that

$$\widehat{\widetilde{R}}_{\widehat{\Gamma}}^{T} \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}} \mathbf{y}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}. \tag{3.6}$$

When using the three-level preconditioner $\widetilde{M}^{-1}$, we do not solve (3.6) exactly. Instead, we replace $\mathbf{y}_{\widehat{\Gamma}}$ by

$$\widetilde{\mathbf{y}}_{\widehat{\Gamma}} = \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^{T} \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}}. \tag{3.7}$$

We will maintain the same relation between $\widetilde{\mathbf{y}}_{\widehat{I}}^{(i)}$ and $\widetilde{\mathbf{y}}_{\widehat{\Gamma}}^{(i)}$, i.e.,

$$\widetilde{\mathbf{y}}_{\widehat{I}}^{(i)} = S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}} \left( \mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)} \widehat{R}_{\widehat{\Gamma}}^{(i)} \widetilde{\mathbf{y}}_{\widehat{\Gamma}} \right). \tag{3.8}$$

**4. Some auxiliary results.** In this section, we will collect a number of results which are needed in our theory. In order to avoid a proliferation of constants, we will use the notation $A \approx B$. This means that there are two constants $c$ and $C$, independent of any parameters, such that $cA \le B \le CA$, where $C < \infty$ and $c > 0$. For the definition of discrete harmonic functions, see [10, Section 4.4].

LEMMA 4.1. *Let $\mathcal{D}$ be a cube with vertices $A_1 = (0,0,0)$, $B_1 = (H,0,0)$, $C_1 = (H,H,0)$, $D_1 = (0,H,0)$, $A_2 = (0,0,H)$, $B_2 = (H,0,H)$, $C_2 = (H,H,H)$ and $D_2 = (0,H,H)$ with a quasi-uniform triangulation of mesh size $h$. Then, there exists a discrete harmonic function $v$ defined in $\mathcal{D}$ such that $\bar{v}_{A_1B_1} \approx 1 + \log \frac{H}{h}$, where $\bar{v}_{A_1B_1}$ is the average of $v$ over the edge $A_1B_1$, $|v|^2_{H^1(\mathcal{D})} \approx H\left(1 + \log \frac{H}{h}\right)$, and $v$ has a zero average over the other edges.*

*Proof:* This lemma follows from a result by Brenner and He [1, Lemma 4.2]: let $N$ be an integer and $G_N$ the function defined on $(0,1)$ by

$$G_N(x) = \sum_{n=1}^{N} \left( \frac{1}{4n-3} \sin\left((4n-3)\pi x\right) \right).$$

$G_N(x)$ is even with respect to the midpoint of $(0,1)$, where it attains its maximum in absolute value. Moreover, we have:

$$|G_N|^2_{H^{1/2}_{00}(0,1)} \approx 1 + \log N \text{ and } \|G_N\|_{L^2(0,1)} \approx 1;$$

see [1, Lemma 3.7].

Let $[-H,0]$ and $[0,H]$ have a mesh inherited from the quasi-uniform meshes on $D_1A_1$ and $A_1B_1$, respectively, and let $g_h(x)$ be the nodal interpolation of $G_N(\frac{x+H}{2H})$. Then, we have $\|g_h\|_{L^\infty(-H,H)} \approx 1 + \log \frac{H}{h}$,

$$|g_h|^2_{H^{1/2}_{00}(-H,H)} \approx 1 + \log \frac{H}{h} \text{ and } \|g_h\|_{L^2(-H,H)} \approx H; \tag{4.1}$$

see [1, Lemma 3.7] or [11, Lemma 1].

Let $\tau_h(x)$ be a function on $[0,H]$ defined as follows:

$$\tau_h(x) = \begin{cases} \frac{x}{h_1} & 0 \le x \le h_1, \\ 1 & h_1 \le x \le H - h_2, \\ \frac{H-x}{h_2} & H - h_2 \le x \le H, \end{cases}$$

where $h_1$ and $h_2$ are the lengths of the two end intervals.

Then the following estimates hold:

$$\|\tau_h\|^2_{L^2(0,H)} \approx H \text{ and } |\tau_h|^2_{H^{1/2}_{00}(0,H)} \approx 1 + \log \frac{H}{h}; \tag{4.2}$$

see [1, Lemma 3.6].

Define the discrete harmonic function $v$ as 0 on the boundary of $\mathcal{D}$ except two open faces $A_1B_1C_1D_1$ and $A_1B_1B_2A_2$. It is defined on these two faces by

$$v(x_1, x_2, 0) = g_h(x_2)\tau_h(x_1), \qquad \text{for } (x_1, x_2) \in A_1B_1C_1D_1,$$

$$v(x_1,0,x_3) = g_h(-x_3)\tau_h(x_1), \qquad \text{for} \ \ (x_1,x_3) \in A_1B_1B_2A_2.$$

It is clear that $\bar{v}_{A_1B_1} \approx 1 + \log\frac{H}{h}$ and that $v$ has a zero average over the other edges. Since $v$ is discrete harmonic in $\mathcal{D}$, we have,

$$\begin{aligned} &|v|^2_{H^1(\mathcal{D})} = |v|^2_{H^{1/2}(\partial\mathcal{D})} \\ \approx\ & |g_h|^2_{H^{1/2}_{00}(-H,H)}\|\tau_h\|^2_{L^2(0,H)} + |\tau_h|^2_{H^{1/2}_{00}(0,H)}\|g_h\|^2_{L^2(-H,H)} \\ \approx\ & H\left(1+\log\frac{H}{h}\right), \end{aligned}$$

where we have used (4.1), (4.2), and [1, Corollary 3.5].

□

*Remark: In Lemma 4.1, we have constructed the function $v$ for a cube $\mathcal{D}$. By using similar ideas, we can construct functions $v$ for other shape-regular polyhedra which will satisfy the same properties and bounds.*

LEMMA 4.2. *Let $\Omega^i_j$ be the subdomains in a subregion $\Omega^i$, $j=1,\cdots,N_i$, and $V^h_{i,j}$ be the standard continuous piecewise trilinear finite element function space in the subdomain $\Omega^i_j$ with a quasi-uniform fine mesh with mesh size $h$. Denote by $\mathcal{E}_k$, $k=1\cdots K_j$, the edges of the subdomain $\Omega^i_j$. Given the average values of $u$, $\bar{u}_{\mathcal{E}_k}$, over each edge, let $u \in V^h_{i,j}$ be the discrete $V^h_{i,j}$-harmonic extension in each subdomain $\Omega^i_j$ with the average values given on the edges of $\Omega^i_j$, $j=1,\cdots,N_i$. Then, there exist two positive constants $C_1$ and $C_2$, which are independent of $\hat{H}$, $H$, and $h$, such that*

$$\begin{aligned} C_1\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}|u|^2_{H^1(\Omega^i_j)}\right) &\le \sum_{j=1}^{N_i}\sum_{k_1,k_2=1}^{K_j} H|\bar{u}_{\mathcal{E}_{k_1}} - \bar{u}_{\mathcal{E}_{k_2}}|^2 \\ &\le C_2\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}|u|^2_{H^1(\Omega^i_j)}\right). \end{aligned}$$

*Proof:* Without loss of generality, we assume that the subdomains are hexahedral. Denote the edges of the subdomain $\Omega^i_j$ by $\mathcal{E}_k$, $k=1,\cdots,12$, and denote the average values of $u$ over these twelve edges by $\bar{u}_{\mathcal{E}_k}$, $k=1,\cdots,12$, respectively.

According to Lemma 4.1, we can construct eleven discrete harmonic functions $\phi_m$, $m=2,\cdots,12$, on $\Omega^i_j$ such that

$$(\overline{\phi}_m)_{\mathcal{E}_k} = \begin{cases} (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})\,(1+\log\frac{H}{h}) & m=k, \\ 0 & m\neq k, \end{cases}$$

and with

$$|\phi_m|^2_{H^1(\Omega^i_j)} \approx (\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})^2\, H(1+\log\frac{H}{h}), \quad m=2,\cdots,12. \tag{4.3}$$

Let $v_j = \frac{1}{1+\log\frac{H}{h}}\left(\sum_{m=2}^{12}\phi_m\right) + \bar{u}_{\mathcal{E}_1}$; we then have $(\bar{v}_j)_{\mathcal{E}_k} = \bar{u}_{\mathcal{E}_k}$, for $k=1,\cdots,12$, and

$$
\begin{aligned}
&|v_j|^2_{H^1(\Omega^i_j)} = |\frac{1}{1+\log\frac{H}{h}}\left(\sum_{m=2}^{12}\phi_m\right) + \bar{u}_{\mathcal{E}_1}|^2_{H^1(\Omega^i_j)} \\
&= \left(\frac{1}{1+\log\frac{H}{h}}\right)^2 |\sum_{m=2}^{12}\phi_m|^2_{H^1(\Omega^i_j)} \le 11\left(\frac{1}{1+\log\frac{H}{h}}\right)^2 \sum_{m=2}^{12}|\phi_m|^2_{H^1(\Omega^i_j)} \\
&\le \left(\frac{1}{C_1^{1/2}(1+\log\frac{H}{h})}\right)^2 H\left(1+\log\frac{H}{h}\right)\sum_{m=2}^{12}(\bar{u}_{\mathcal{E}_m} - \bar{u}_{\mathcal{E}_1})^2 \\
&\le \frac{1}{C_1(1+\log\frac{H}{h})}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.
\end{aligned}
$$

Here, we have used (4.3) for the penultimate inequality.

By the definition of $u$, we have,

$$
|u|^2_{H^1(\Omega^i_j)} \le |v_j|^2_{H^1(\Omega^i_j)} \le \frac{1}{C_1(1+\log\frac{H}{h})}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.
$$

Summing over all the subdomains in the subregion $\Omega^i$, we have,

$$
C_1\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}|u|^2_{H^1(\Omega^i_j)}\right) \le \sum_{j=1}^{N_i}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2.
$$

This proves the first inequality.

We prove the second inequality as follows:

$$
\begin{aligned}
&\sum_{j=1}^{N_i}\sum_{k=1}^{12} H(\bar{u}_{\mathcal{E}_k} - \bar{u}_{\mathcal{E}_1})^2 = \sum_{j=1}^{N_i}\sum_{k=1}^{12} H|\overline{(u-\bar{u}_{\mathcal{E}_1})}_{\mathcal{E}_k}|^2 \\
&\le C_2\left(\sum_{j=1}^{N_i} H\frac{1}{H}\|u-\bar{u}_{\mathcal{E}_1}\|^2_{L^2(\mathcal{E}_k)}\right) \le C_2\left(\sum_{j=1}^{N_i}(1+\log\frac{H}{h})|u|^2_{H^1(\Omega^i_j)}\right) \\
&\le C_2\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}|u|^2_{H^1(\Omega^i_j)}\right).
\end{aligned}
$$

Here, we have used a standard finite element Sobolev inequality, see [10, Lemma 4.30] for the second inequality and [10, Lemma 4.16] for the penultimate inequality.

We complete the proof of the second inequality by using the triangle inequality.

□

We now introduce a new mesh on each subregion; we follow [2, 9]. The purpose for introducing this new mesh is to relate the quadratic form of Lemma 4.2 to one for a more conventional finite element space.

Given a subregion $\Omega^i$ and subdomains $\Omega^i_j$, $j = 1, \cdots, N_i$, let $\mathcal{T}$ be a quasi-uniform sub-triangulation of $\Omega^i$ such that its set of the vertices include the vertices and the midpoints of edges of $\Omega^i_j$. For the hexahedral case, we decomposed each hexahedron into 8 hexahedra by connecting the midpoints of edges. We then partition the vertices in the new mesh $\mathcal{T}$ into two sets. The midpoints of edges are called primal and the others are called secondary. We call two vertices in the triangulation $\mathcal{T}$ adjacent if there is an edge of $\mathcal{T}$ between them, as in the standard finite element context.

Let $U_H(\Omega)$ be the continuous piecewise trilinear finite element function space with respect to the new triangulation $\mathcal{T}$. For a subregion $\Omega^i$, $U_H(\Omega^i)$ and $U_H(\partial\Omega^i)$ are defined as restrictions:

$$U_H(\Omega^i) = \{u|_{\Omega^i} :\ u \in U_H(\Omega)\}, \qquad U_H(\partial\Omega^i) = \{u|_{\partial\Omega^i} :\ u \in U_H(\Omega)\}.$$

We define a mapping $I_H^{\Omega^i}$ of any function $\phi$, defined at the primal vertices in $\Omega^i$, to $U_H(\Omega^i)$ by

(4.4)

$$I_H^{\Omega^i}\phi(x) = \begin{cases} \phi(x),\ \text{if } x \text{ is a primal node;} \\ \\ \text{the average of the values at all adjacent primal nodes} \\ \text{on the edges of } \Omega^i,\ \text{if } x \text{ is a vertex of } \Omega^i; \\ \\ \text{the average of the values at two adjacent primal nodes} \\ \text{on the same edge of } \Omega^i,\ \text{if } x \text{ is an edge secondary node of } \Omega^i; \\ \\ \text{the average of the values at all adjacent primal nodes on the} \\ \ \text{boundary of } \Omega^i,\ \text{if } x \text{ is a face secondary boundary node of } \Omega^i; \\ \\ \text{the average of the values at all adjacent primal nodes} \\ \ \text{if } x \text{ is a interior secondary node of } \Omega^i; \\ \\ \text{the result of trilinear interpolation using the vertex values,} \\ \ \text{if } x \text{ is not a vertex of } \mathcal{T}. \end{cases}$$

We recall that $W_c^{(i)}$ is the discrete space of the values at the primal nodes given by the subdomain edge average values. $I_H^{\Omega^i}$ can be considered as a map from $W_c^{(i)}$ to $U_H(\Omega^i)$ or as a map from $U_H(\Omega^i)$ to $U_H(\Omega^i)$.

Let $I_H^{\partial\Omega^i}$ be the mapping of a function $\phi$ defined at the primal vertices on the boundary of $\Omega^i$ to $U_H(\partial\Omega^i)$ and defined by $I_H^{\partial\Omega^i}\phi = (I_H^{\Omega^i}\phi_e)|_{\partial\Omega^i}$, where $\phi_e$ is any function in $W_c^{(i)}$ such that $\phi_e|_{\partial\Omega^i} = \phi$. The map is well defined since the boundary values of $I_H^{\Omega^i}\phi_e$ only depend on the boundary values of $\phi_e$.

Finally, let

$$\widetilde{U}_H(\Omega^i) = \{\psi = I_H^{\Omega^i}\phi,\ \phi \in U_H(\Omega^i)\}, \qquad \widetilde{U}_H(\partial\Omega^i) = \{\psi|_{\partial\Omega^i},\ \psi \in \widetilde{U}_H(\Omega^i)\}.$$

$I_H^{\partial\Omega^i}$ also can be considered as a map from $W_{c,\widehat{\Gamma}}^{(i)}$ to $\widetilde{U}_H(\partial\Omega^i)$.

*Remark: We carefully define the* $I_H^{\Omega^i}$ *and* $I_H^{\partial\Omega^i}$ *so that, if the edge averages of* $w_i \in W_{c,\widehat{\Gamma}}^{(i)}$ *and* $w_j \in W_{c,\widehat{\Gamma}}^{(j)}$ *over an edge* $\mathcal{E}$ *are the same, we have* $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}}$. *Here we need to use a weighted average which has a smaller weight at the two end points. But this will not effect our analysis. We could also define a weighted edge average of* $w_i$ *and* $w_j$ *and obtain* $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}}$ *for the usual average.*

We list some useful lemmas from [2]. For the proofs of Lemma 4.3 and Lemma 4.4, see [2, Lemma 6.1 and Lemma 6.2], respectively.

LEMMA 4.3. *There exists a constant $C > 0$, independent of $H$ and $|\Omega^i|$, the volume of $\Omega^i$, such that*

$$|I_H^{\Omega^i}\phi|_{H^1(\Omega^i)} \leq C|\phi|_{H^1(\Omega^i)} \text{ and } \|I_H^{\Omega^i}\phi\|_{L^2(\Omega^i)} \leq C\|\phi\|_{L^2(\Omega^i)}, \quad \forall \phi \in U_H(\Omega^i).$$

LEMMA 4.4. *For $\hat{\phi} \in \widetilde{U}_H(\partial\Omega^i)$,*

$$\inf_{\phi \in \widetilde{U}_H(\Omega^i), \phi|_{\partial\Omega^i} = \hat{\phi}} \|\phi\|_{H^1(\Omega^i)} \approx \|\hat{\phi}\|_{H^{1/2}(\partial\Omega^i)},$$

$$\inf_{\phi \in \widetilde{U}_H(\Omega^i), \phi|_{\partial\Omega^i} = \hat{\phi}} |\phi|_{H^1(\Omega^i)} \approx |\hat{\phi}|_{H^{1/2}(\partial\Omega^i)}.$$

LEMMA 4.5. *There exist constants $C_1$ and $C_2 > 0$, independent of $\hat{H}$, $H$, $h$, and the coefficient of (2.1) such that for all $w_i \in W_{c,\widehat{\Gamma}}^{(i)}$,*

$$\rho_i C_1 |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} \leq \left(1 + \log\frac{H}{h}\right)(T^{(i)}w_i, w_i) \leq \rho_i C_2 |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)},$$

*where $(T^{(i)}w_i, w_i) = w_i^T T^{(i)} w_i = |w_i|^2_{T^{(i)}}$ and $T^{(i)} = S^{(i)}_{\Pi_{\widehat{\Gamma}\widehat{\Gamma}}} - S^{(i)}_{\Pi_{\widehat{\Gamma}\widehat{I}}} S^{(i)^{-1}}_{\Pi_{\widehat{I}\widehat{I}}} S^{(i)^T}_{\Pi_{\widehat{\Gamma}\widehat{I}}}$.*

*Proof:* By the definition of $T^{(i)}$, we have

$$\begin{aligned}
&\left(1 + \log\frac{H}{h}\right)(T^{(i)}w_i, w_i) = \left(1 + \log\frac{H}{h}\right) \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^i} = w_i} |v|^2_{S_\Pi^{(i)}} \\
&= \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^i} = w_i} \rho_i \left(1 + \log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i} \inf_{u \in V_{i,j}^h, \bar{u}_{\mathcal{E}} = v, \mathcal{E} \subset \partial\Omega_j^i} |u|^2_{H^1(\Omega_j^i)}\right) \\
&\approx \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^i} = w_i} \rho_i \sum_{j=1}^{N_i} \sum_{k_1, k_2 = 1}^{K_j} H|\bar{v}_{\mathcal{E}_{k_1}} - \bar{v}_{\mathcal{E}_{k_2}}|^2 \\
&\approx \inf_{v \in W_c^{(i)}, v|_{\partial\Omega^i} = w_i} \rho_i |I_H^{\Omega^i} v|^2_{H^1(\Omega^i)} \approx \inf_{\phi \in \widetilde{U}_H(\Omega^i), \phi|_{\partial\Omega^i} = I_H^{\partial\Omega^i} w_i} \rho_i |\phi|^2_{H^1(\Omega^i)} \\
&\approx \rho_i |I_H^{\partial\Omega^i} w_i|_{H^{1/2}(\partial\Omega^i)}.
\end{aligned}$$

We use Lemma 4.2 for the third bound, the definitions of $I_H^{\Omega^i}$ and $I_H^{\partial\Omega^i}$ for the fourth and fifth bounds, and Lemma 4.4 for the final one.

□

To be fully rigorous, we assume that there is a quasi-uniform coarse triangulation of each subregion. We can then obtain uniform constants $C_1$ and $C_2$ in Lemma 4.5, which work for all the subregions.

We define the interface averages operator $\widehat{E}_{\widehat{D}}$ on $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$ as $\widehat{E}_{\widehat{D}} = \widehat{\widetilde{R}}_{\widehat{\Gamma}} \widehat{\widetilde{R}}^T_{\widehat{D},\widehat{\Gamma}}$,which computes the averages across the subregion interface $\widehat{\Gamma}$ and then distributes the averages to the boundary points of the subregions.

The interface average operator $\widehat{E}_{\widehat{D}}$ has the following property:

LEMMA 4.6.

$$|\widehat{E}_{\widehat{D}} \mathbf{w}_{\widehat{\Gamma}}|^2_{\widehat{T}} \leq C\left(1 + \log\frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widehat{T}},$$

*for any $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, where $C$ is a positive constant independent of $\hat{H}$, $H$, $h$, and the coefficients of (2.1). Here $\widetilde{T}$ is define in (3.4).*

*Proof:* Let $w_i = \widehat{\widehat{R}}^{(i)}_{\widehat{\Gamma}} \mathbf{w}_{\widehat{\Gamma}} \in W^{(i)}_{c,\widehat{\Gamma}}$. We rewrite the formula for $v := w_{\widehat{\Gamma}} - \widehat{E}_{\widehat{\Gamma}} w_{\widehat{\Gamma}}$ for an arbitrary element $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, and find that for $i = 1, \cdots, N$,

$$v_i(x) := (w_{\widehat{\Gamma}}(x) - \widehat{E}_{\widehat{\Gamma}} w_{\widehat{\Gamma}}(x))_i = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger (w_i(x) - w_j(x)), \quad x \in \partial\Omega^i \cap \widehat{\Gamma}. \tag{4.5}$$

Here $\mathcal{N}_x$ is the set of indices of the subregions that have $x$ on their boundaries.

We have

$$|\widehat{E}_{\widehat{D}} \mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} = \sum_{i=1}^N |w_i - v_i|^2_{T^{(i)}} \leq 2\sum_{i=1}^N |w_i|^2_{T^{(i)}} + 2\sum_{i=1}^N |v_i|^2_{T^{(i)}} \text{ and } |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} = \sum_{i=1}^N |w_i|^2_{T^{(i)}}.$$

We can therefore focus on the estimate of the contribution from a single subregion $\Omega^i$ and proceed as in the proof of [10, Lemma 6.36].

We will also use the simple inequality

$$\rho_i {\delta_j^\dagger}^2 \leq \min(\rho_i, \rho_j), \text{ for } \gamma \in [1/2, \infty). \tag{4.6}$$

By Lemma 4.5,

$$(T^{(i)} v_i, v_i) \leq C_2 \frac{1}{(1 + \log \frac{H}{h})} \rho_i |I_H^{\partial\Omega^i}(v_i)|^2_{H^{1/2}(\partial\Omega^i)}. \tag{4.7}$$

Let $L_i = I_H^{\partial\Omega^i}(v_i)$. We have, by using a partition of unity as in [10, Lemma 6.36],

$$L_i = \sum_{\mathcal{F} \subset \partial\Omega_i} I^H(\theta_{\mathcal{F}} L_i) + \sum_{\mathcal{E} \subset \partial\Omega_i} I^H(\theta_{\mathcal{E}} L_i) + \sum_{\mathcal{V} \in \partial\Omega_i} \theta_{\mathcal{V}} L_i(\mathcal{V}),$$

where $I^H$ is the nodal piecewise linear interpolant on the coarse mesh $\mathcal{T}$. We note that the analysis for face and edge terms is almost identical to that in [10, Lemma 6.36]. But the vertex terms are different because of $I_H^{\partial\Omega^i}$. We only need to consider the vertex term when two subregion share at least an edge. This make the analysis simpler than in the proof of [10, Lemma 6.36].

**Face Terms.** First consider,

$$I^H(\theta_{\mathcal{F}} L_i) = I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^i}(\delta_j^\dagger (w_i - w_j))).$$

Similar to [10, Lemma 6.36], we obtain, by using (4.6),

(4.8)
$$\begin{aligned}
&\rho_i |I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^i}(\delta_j^\dagger (w_i - w_j)))|^2_{H^{1/2}(\partial\Omega^i)} \\
&= \rho_i {\delta_j^\dagger}^2 |I^H(\theta_{\mathcal{F}} I_H^{\partial\Omega^i}(w_i - w_j))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\leq \min(\rho_i, \rho_j) |I^H(\theta_{\mathcal{F}}((I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}) - (I_H^{\partial\Omega^i} w_j - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}}) + \\
&((\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}})))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\leq 3\min(\rho_i, \rho_j) \Big( |I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} + \\
&\quad |I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_j - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} + \\
&\quad |\theta_{\mathcal{F}}((\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}})|^2_{H^{1/2}(\partial\Omega^i)} \Big).
\end{aligned}$$

By the definition of $I_H^{\partial\Omega^i}$,

$$I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_j)) = I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^j} w_j)) \quad \text{and} \quad (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}.$$

By [10, Lemma 4.26], the first and second terms in (4.8) can be estimated as follows:

$$\begin{aligned}
&\min(\rho_i,\rho_j)(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} + \\
&|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_j - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)}) \\
=\ &\min(\rho_i,\rho_j)(|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)} + \\
&|I^H(\theta_{\mathcal{F}}(I_H^{\partial\Omega^j} w_j - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}))|^2_{H^{1/2}(\partial\Omega^i)}) \\
\leq\ & C\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} + \rho_j |I_H^{\partial\Omega^j} w_j|^2_{H^{1/2}(\partial\Omega^j)}\right).
\end{aligned}$$

Let $\mathcal{E} \subset \partial\mathcal{F}$. Since the edge averages of $w_i$ and $w_j$ are the same, by the definition of $I_H^{\partial\Omega^i}$ and $I_H^{\partial\Omega^j}$, we have $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}}$. As we have pointed out before, we use the weighted average which has a smaller weight at the two end points.

We then have

$$\begin{aligned}
&|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}|^2 \\
\leq\ & 2\left(|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2 + |(\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{F}}|^2\right).
\end{aligned} \tag{4.9}$$

It is sufficient to consider the first term on the right hand side. Using [10, Lemma 4.30], we find

$$\begin{aligned}
&|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2 \\
=\ & |(\overline{I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}})_{\mathcal{E}}|^2 \leq C/\hat{H}_i \| I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} \|^2_{L^2(\mathcal{E})},
\end{aligned}$$

and, by using [10, Lemma 4.17] and the Poincaré inequality given as [10, Lemma A.17],

$$|(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2 \leq C/\hat{H}_i \left(1+\log\frac{\hat{H}}{H}\right) |I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}}|^2_{H^{1/2}(\mathcal{F})}.$$

Combining this with the bound for $\theta_{\mathcal{F}}$ in [10, Lemma 4.26], we have:

$$\begin{aligned}
&\min(\rho_i,\rho_j)|\theta_{\mathcal{F}}((\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{F}} - (\overline{I_H^{\partial\Omega^i} w_j})_{\mathcal{F}})|^2_{H^{1/2}(\partial\Omega^i)} \\
\leq\ & C\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} + \rho_j |I_H^{\partial\Omega^j} w_j|^2_{H^{1/2}(\partial\Omega^j)}\right).
\end{aligned}$$

**Edge Terms.** We can develop the same estimate as in [10, Lemma 6.34]. For simplicity, we only consider an edge $\mathcal{E}$ common to four subregions $\Omega^i$, $\Omega^j$, $\Omega^k$, and $\Omega^l$.

$$\begin{aligned}
&\rho_i |I^H(\theta_{\mathcal{E}} L_i)|^2_{H^{1/2}(\partial\Omega^i)} \\
\leq\ & \rho_i \Big(|I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta_j^\dagger(w_i - w_j)))|^2_{H^{1/2}(\partial\Omega^i)} + \\
&|I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta_k^\dagger(w_i - w_k)))|^2_{H^{1/2}(\partial\Omega^i)} + |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta_l^\dagger(w_i - w_l)))|^2_{H^{1/2}(\partial\Omega^i)}\Big).
\end{aligned} \tag{4.10}$$

We recall that $\delta_j^\dagger$, $\delta_k^\dagger$, and $\delta_l^\dagger$ are constants.

By the definition of $I_H^{\partial\Omega^i}$, $I_H^{\partial\Omega^j}$, $I_H^{\partial\Omega^k}$, and $I_H^{\partial\Omega^l}$, we have

$$\theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_j) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^j} w_j), \quad \theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_k) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^k} w_k), \quad \theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_l) = \theta_{\mathcal{E}}(I_H^{\partial\Omega^l} w_l),$$

and,

$$(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}} = (\overline{I_H^{\partial\Omega^l} w_l})_{\mathcal{E}}.$$

We assume that $\Omega^i$ shares a face with $\Omega^j$ as well as $\Omega^l$, and shares an edge only with $\Omega^k$.

We consider the second term in (4.10) first. By [10, Lemmas 4.19 and 4.17], and (4.6), we have

$$\begin{aligned}
&\rho_i |I^H(\theta_{\mathcal{E}} I_H^{\partial\Omega^i}(\delta_k^\dagger(w_i - w_k)))|^2_{H^{1/2}(\partial\Omega^i)} \\
&\leq C\rho_i {\delta_k^\dagger}^2 \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}}) - \theta_{\mathcal{E}}(I_H^{\partial\Omega^k} w_k - (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})} \\
&\leq 2C \Big(\rho_i \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})} + \\
&\quad \rho_k \|I^H(\theta_{\mathcal{E}}(I_H^{\partial\Omega^k} w_k - (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}}))\|^2_{L^2(\mathcal{E})}\Big) \\
&\leq 2C \left(\rho_i \|I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}}\|^2_{L^2(\mathcal{E})} + \rho_k \|I_H^{\partial\Omega^k} w_k - (\overline{I_H^{\partial\Omega^k} w_k})_{\mathcal{E}}\|^2_{L^2(\mathcal{E})}\right) \\
&\leq 2C \left(1 + \log \frac{\hat{H}}{H}\right) \left(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\mathcal{F}^i)} + \rho_k |I_H^{\partial\Omega^k} w_k|^2_{H^{1/2}(\mathcal{F}^k)}\right) \\
&\leq 2C \left(1 + \log \frac{\hat{H}}{H}\right) \left(\rho_i |I_H^{\partial\Omega^i} w_i|^2_{H^{1/2}(\partial\Omega^i)} + \rho_k |I_H^{\partial\Omega^k} w_k|^2_{H^{1/2}(\partial\Omega^k)}\right),
\end{aligned}$$

where $\mathcal{F}^i$ is a face of $\Omega^i$ and $\mathcal{F}^k$ is a face of $\Omega^k$, and $\mathcal{F}^i$ and $\mathcal{F}^k$ share the edge $\mathcal{E}$.

The first term and the third term can be estimated similarly.

**Vertex Terms.** We can do the estimate similarly to that of the proof in [10, Lemma 6.36]. We have

$$\rho_i |\theta_{\mathcal{V}} L_i(\mathcal{V})|^2_{H^{1/2}(\partial\Omega_i)} = \rho_i |\theta_{\mathcal{V}}(I_H^{\partial\Omega^i} v_i)(\mathcal{V})|^2_{H^{1/2}(\partial\Omega_i)}. \tag{4.11}$$

By (4.5) and the definition of $I_H^{\partial\Omega^i}$, we see that $(I_H^{\partial\Omega^i} v_i)(\mathcal{V})$ is nonzero only when two subregions share one or several edges with a common vertex $\mathcal{V}$.

In the definition of $I_H^{\Omega^i}$, we denote by $\mathcal{E}_{i,m}$, $m = 1, 2, 3\cdots$, the edges in $\Omega^i$ which share $\mathcal{V}$. Denote by $v_{i,m}$ the primal nodes on the edges $\mathcal{E}_{i,m}$ which are adjacent to $\mathcal{V}$.

By the definition of $I_H^{\Omega^i}$, (4.11), and $|\theta_{\mathcal{V}}|^2_{H^{1/2}(\partial\Omega^i)} \leq CH_i$, we have,

$$\begin{aligned}
\rho_i |\theta_{\mathcal{V}}(I_H^{\partial\Omega^i} v_i)(\mathcal{V})|^2_{H^{1/2}(\partial\Omega^i)} &\leq C\rho_i |\sum_m v_i(v_{i,m})|^2 |\theta_{\mathcal{V}}|^2_{H^{1/2}(\partial\Omega^i)} \\
&\leq C\rho_i H_i \sum_m |v_i(v_{i,m})|^2.
\end{aligned} \tag{4.12}$$

Let us look at the first term in (4.12), the other terms can be estimated in the same way.

$$
\begin{aligned}
&\rho_i H_i |v_i(v_{i,1})|^2 \\
&= \rho_i H_i | \sum_{j, \mathcal{E}_{i,1} \subset \Omega_j} \delta_j^\dagger (w_i(v_{i,1}) - w_j(v_{i,1}))|^2 \\
&\leq C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^j} \min(\rho_i, \rho_j) H_i |w_i(v_{i,1}) - w_j(v_{i,1}))|^2 \\
&= C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^j} \min(\rho_i, \rho_j) H_i |I_H^{\partial\Omega^i} w_i(v_{i,1}) - I_H^{\partial\Omega^j} w_j(v_{i,1})|^2 \\
&\leq C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^j} \min(\rho_i, \rho_j) H_i \left( |I_H^{\partial\Omega^i} w_i(v_{i,1}) - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}}|^2 + \right. \\
&\quad \left. |I_H^{\partial\Omega^j} w_j(v_{i,1}) - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}_{i,1}}|^2 \right) \\
&\leq C \sum_{j, \mathcal{E}_{i,1} \subset \Omega^j} \min(\rho_i, \rho_j) \left( H_i | \left( I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}} \right) (v_{i,1})|^2 + \right. \\
&\quad \left. H_i | \left( I_H^{\partial\Omega^j} w_j - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}i,1} \right) (v_{i,1})|^2 \right) \\
&\leq C \sum_{j, \mathcal{E}i,1 \subset \Omega^j} \min(\rho_i, \rho_j) \left( \| I_H^{\partial\Omega^i} w_i - (\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}} \|_{L^2(\mathcal{E}_{i,1})}^2 + \right. \\
&\quad \left. \| I_H^{\partial\Omega^j} w_j - (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}_{i,1}} \|_{L^2(\mathcal{E}_{i,1})}^2 \right) \\
&\leq C \sum_{j, \mathcal{E}i,1 \subset \Omega^j} \left( 1 + \log \frac{\hat{H}}{H} \right) \left( \rho_i |I_H^{\partial\Omega^i} w_i|_{H^{1/2}(\partial\Omega^i)}^2 + \rho_j |I_H^{\partial\Omega^j} w_j|_{H^{1/2}(\partial\Omega^i)}^2 \right).
\end{aligned}
$$

For the third equality, we use here that $v_{i,1}$ is a primary node. For the fourth inequality, we use that $(\overline{I_H^{\partial\Omega^i} w_i})_{\mathcal{E}_{i,1}} = (\overline{I_H^{\partial\Omega^j} w_j})_{\mathcal{E}_{i,1}}$. We use [10, Lemmas B.5] for the sixth inequality and [10, Lemma 4.17] for the last inequality.

Combining all face, edge, and vertex terms, we obtain

$$
\rho_i |I_H^{\partial\Omega^i}(v_i)|_{H^{1/2}(\partial\Omega^i)}^2 \leq C \left( 1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j: \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} \rho_j |I_H^{\partial\Omega^i}(w_j)|_{H^{1/2}(\partial\Omega^j)}^2. \tag{4.13}
$$

Using (4.13), Lemma 4.5 and (4.7), we obtain

$$
\begin{aligned}
(T^{(i)} v_i, v_i) = |v_i|_{T^{(i)}}^2 &\leq C_2 \frac{1}{(1 + \log \frac{H}{h})} \rho_i |I_H^{\partial\Omega^i}(v_i)|_{H^{1/2}(\partial\Omega^i)}^2 \\
&\leq C C_2 \frac{\left( 1 + \log \frac{\hat{H}}{H} \right)^2}{(1 + \log \frac{H}{h})} \sum_{j: \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} \rho_j |I_H^{\partial\Omega^i}(w_j)|_{H^{1/2}(\partial\Omega^j)}^2 \\
&\leq C \frac{C_2}{C_1} \left( 1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j: \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} (T^{(j)} w_j, w_j) \\
&= C \frac{C_2}{C_1} \left( 1 + \log \frac{\hat{H}}{H} \right)^2 \sum_{j: \partial\Omega^j \cap \partial\Omega^i \neq \emptyset} |w_j|_{T^{(j)}}^2.
\end{aligned}
$$

□

Lemma 4.7. *Given any* $\mathbf{u}_{\Gamma} \in \widehat{\mathbf{W}}_{\Gamma}$, *let* $\mathbf{\Psi} = \Phi^T \widetilde{R}_{D,\Gamma}\mathbf{u}_{\Gamma}$. *We have,*

$$\mathbf{\Psi}^T S_{\Pi}^{-1}\mathbf{\Psi} \leq \mathbf{\Psi}^T \widetilde{S}_{\Pi}^{-1}\mathbf{\Psi}^T \leq C\left(1+\log\frac{\hat{H}}{H}\right)^2 \mathbf{\Psi}^T S_{\Pi}^{-1}\mathbf{\Psi}.$$

*Proof:* Using (3.3), (3.5), and (3.6), we have

(4.14)

$$\begin{aligned}
&\mathbf{\Psi}^T S_{\Pi}^{-1}\mathbf{\Psi} = \sum_{i=1}^{N} \Psi_{\widehat{I}}^{(i)^T}\mathbf{y}_{\widehat{I}}^{(i)} + \mathbf{\Psi}_{\widehat{\Gamma}}^T\mathbf{y}_{\widehat{\Gamma}}\\
&= \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\left(S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)}\widehat{R}_{\widehat{\Gamma}}^{(i)}\mathbf{y}_{\widehat{\Gamma}})\right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^{N}\widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}I}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}^{(i)}\right)^T \mathbf{y}_{\widehat{\Gamma}}\\
&= \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\mathbf{y}_{\widehat{\Gamma}} = \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}}.
\end{aligned}$$

Using (3.8), (3.5), and (3.7), we also have

$$\begin{aligned}
&\mathbf{\Psi}^T \widetilde{S}_{\Pi}^{-1}\mathbf{\Psi} = \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\widetilde{\mathbf{y}}_{\widehat{I}^{(i)}} + \mathbf{\Psi}_{\widehat{\Gamma}}^T\widetilde{\mathbf{y}}_{\widehat{\Gamma}}\\
&= \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T}\left(S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}(\mathbf{\Psi}_{\widehat{I}}^{(i)} - S_{\Pi_{\widehat{I}\widehat{\Gamma}}}^{(i)}\widehat{R}_{\widehat{\Gamma}}^{(i)}\widetilde{\mathbf{y}}_{\widehat{\Gamma}})\right) + \left(\mathbf{h}_{\widehat{\Gamma}} + \sum_{i=1}^{N}\widehat{R}_{\widehat{\Gamma}}^{(i)^T} S_{\Pi_{\widehat{\Gamma}\widehat{I}}}^{(i)} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}^{(i)}\right)^T \widetilde{\mathbf{y}}_{\widehat{\Gamma}}\\
&= \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\widetilde{\mathbf{y}}_{\widehat{\Gamma}} = \sum_{i=1}^{N} \mathbf{\Psi}_{\widehat{I}}^{(i)^T} S_{\Pi_{\widehat{I}\widehat{I}}}^{(i)^{-1}}\mathbf{\Psi}_{\widehat{I}}^{(i)} + \mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right)\mathbf{h}_{\widehat{\Gamma}}.
\end{aligned}$$

We only need to compare $\mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}}$ and $\mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right)\mathbf{h}_{\widehat{\Gamma}}$ for any $\mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{F}}_{c,\widehat{\Gamma}}$. The following estimate is established as [6, Theorem 1]. Let

$$\mathbf{w}_{\widehat{\Gamma}} = \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}} \text{ and } \mathbf{v}_{\widehat{\Gamma}} = \widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\mathbf{h}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}. \tag{4.15}$$

Noting the fact that $\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}} = \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widehat{\widetilde{R}}_{\widehat{\Gamma}} = I$ and using (4.15), we have,

$$\begin{aligned}
&\mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}^T\mathbf{w}_{\widehat{\Gamma}} = \mathbf{h}_{\widehat{\Gamma}}^T\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}\\
&= \mathbf{h}_{\widehat{\Gamma}}^T\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}} = \left(\widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\mathbf{h}_{\widehat{\Gamma}}\right)^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}\\
&= \mathbf{v}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}} = <\mathbf{v}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}>_{\widetilde{T}} \leq <\mathbf{v}_{\widehat{\Gamma}}, \mathbf{v}_{\widehat{\Gamma}}>_{\widetilde{T}}^{1/2} <\widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}, \widehat{\widetilde{R}}_{\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}}>_{\widetilde{T}}^{1/2}\\
&= \left(\mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right)\mathbf{h}_{\widehat{\Gamma}}\right)^{1/2}\left(\mathbf{h}_{\widehat{\Gamma}}^T\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)^{-1}\mathbf{h}_{\widehat{\Gamma}}\right)^{1/2}.
\end{aligned}$$

We obtain that

$$\mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \leq \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}}.$$

On the other hand,

$$\begin{aligned}
&\mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} = \mathbf{w}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right) \left( \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \\
&= < \mathbf{w}_{\widehat{\Gamma}}, \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \left( \widetilde{T}^{-1} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}} \right) >_{\left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)} = < \mathbf{w}_{\widehat{\Gamma}}, \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}} >_{\left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)} \\
&\leq < \mathbf{w}_{\widehat{\Gamma}}, \mathbf{w}_{\widehat{\Gamma}} >^{1/2}_{\left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)} < \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}}, \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}} >^{1/2}_{\left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)} \\
&= \left( \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} < \widetilde{\widehat{R}}_{\widehat{\Gamma}} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}}, \widetilde{\widehat{R}}_{\widehat{\Gamma}} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \mathbf{v}_{\widehat{\Gamma}} >^{1/2}_{\widetilde{T}} \\
&= \left( \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} |\widehat{E}_{\widehat{D}} \mathbf{v}_{\widehat{\Gamma}}|_{\widetilde{T}} \\
&\leq C \left( 1 + \log \frac{\hat{H}}{H} \right) \left( \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} |\mathbf{v}_{\widehat{\Gamma}}|_{\widetilde{T}} \\
&= C \left( 1 + \log \frac{\hat{H}}{H} \right) \left( \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2} \left( \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \right)^{1/2},
\end{aligned}$$

where we use Lemma 4.6 for the penultimate inequality.

Finally we obtain that

$$\mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widetilde{\widehat{R}}_{\widehat{D},\widehat{\Gamma}} \right) \mathbf{h}_{\widehat{\Gamma}} \leq C \left( 1 + \log \frac{\hat{H}}{H} \right)^2 \left( \mathbf{h}_{\widehat{\Gamma}}^T \left( \widetilde{\widehat{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widetilde{\widehat{R}}_{\widehat{\Gamma}} \right)^{-1} \mathbf{h}_{\widehat{\Gamma}} \right).$$

□

**5. Condition number estimate for the new preconditioner.** In order to estimate the condition number for the system with the new preconditioner $\widetilde{M}^{-1}$, we compare it to the system with the preconditioner $M^{-1}$.

LEMMA 5.1. *Given any* $\mathbf{u}_{\boldsymbol{\Gamma}} \in \widehat{\mathbf{W}}_\Gamma$,

$$\mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma \leq \mathbf{u}_\Gamma^T \widetilde{M}^{-1} \mathbf{u}_\Gamma \leq C \left( 1 + \log \frac{\hat{H}}{H} \right)^2 \mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma. \tag{5.1}$$

*Proof:* We have, for any $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$,

$$\begin{aligned}
&\mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma \\
&= \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \sum_{i=1}^N \left( \mathbf{0} \; R_\Delta^{(i)^T} \right) \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta} \right\} \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma \\
&+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi S_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma.
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbf{u}_\Gamma^T \widetilde{M}^{-1} \mathbf{u}_\Gamma \\
&= \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \left\{ R_{\Gamma\Delta}^T \sum_{i=1}^{N} \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta} \right\} \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma \\
&+ \mathbf{u}_\Gamma^T \widetilde{R}_{D,\Gamma}^T \Phi \widetilde{S}_\Pi^{-1} \Phi^T \widetilde{R}_{D,\Gamma} \mathbf{u}_\Gamma .
\end{aligned}
$$

We obtain our result by using Lemma 4.7.

□

THEOREM 5.2. *The condition number for the system with the three-level preconditioner $\widetilde{M}^{-1}$is bounded by $C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$.*

*Proof:* Combining the condition number bound, given in (2.5), for the two-level BDDC method, and Lemma 5.1, we find that the condition number for the three-level method is bounded by $C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$.

□

**6. Using Chebyshev iterations.** Another approach to the three-level BDDC methods is to use an iterative method with a preconditioner to solve (3.6). Here, we consider a Chebyshev method with a fixed number of iterations and use $\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}$ as a preconditioner. Denoting the eigenvalues of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right) \left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)$ by $\lambda_j$, we need two input parameters $l$ and $u$, estimates for the minimum and maximum values of $\lambda_j$, for the Chebyshev iterations. From our analysis above, we know that $l = 1$ and $\max_j \lambda_j \leq C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$. We can use the conjugate gradient method to obtain an estimate for the largest eigenvalue at the beginning of the computation to choose a proper $u$. Let $\alpha = \frac{2}{l+u}$, $\mu = \frac{u+l}{u-l}$, and $\sigma_j = 1 - \alpha\lambda_j$. As for the two dimensional case in [11, Section 6], we have the following theorem. No new ideas are required.

THEOREM 6.1. *The condition number using the three-level preconditioner $\widehat{M}^{-1}$ with $k$ Chebyshev iterations is bounded by $C\frac{C_2(k)}{C_1(k)}(1+\log\frac{H}{h})^2$, where*

$$C_1(k) = \min_j \left(1 - \frac{\cosh(k\cosh^{-1}(\mu\sigma_j))}{\cosh(k\cosh^{-1}(\mu))}\right)$$

$$C_2(k) = \max_j \left(1 - \frac{\cosh(k\cosh^{-1}(\mu\sigma_j))}{\cosh(k\cosh^{-1}(\mu))}\right),$$

*and $\frac{C_2(k)}{C_1(k)} \to 1$ as $k \to \infty$.*

**7. Numerical experiments.** We have applied our two three-level BDDC algorithms to the model problem (2.1), where $\Omega = [0,1]^3$. We decompose the unit cube into $\widehat{N}\times\widehat{N}\times\widehat{N}$ subregions with the side-length $\hat{H} = 1/\widehat{N}$ and each subregion into $N\times N\times N$ subdomains with the side-length $H = \hat{H}/N$. Equation (2.1) is discretized, in each subdomain, by conforming piecewise trilinear elements with an element diameter $h$. The preconditioned conjugate gradient iteration is stopped when the norm of the residual has been reduced by a factor of $10^{-6}$.

Table 1

*Eigenvalue bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subregions, $\frac{\hat{H}}{H}=3$ and $\frac{H}{h}=3$*

| | | Case 1 | | Case 2 |
|---|---|---|---|---|
| Num. of Subregions | Iter. | Cond. # | Iter. | Cond. # |
| $3\times3\times3$ | 9 | 2.6603 | 9 | 2.2559 |
| $4\times4\times4$ | 10 | 2.8701 | 10 | 2.5245 |
| $5\times5\times5$ | 11 | 2.9668 | 11 | 2.8074 |
| $6\times6\times6$ | 11 | 3.0190 | 11 | 2.8477 |

Table 2

*Eigenvalue bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subdomains and the size of subdomain problems with $3\times3\times3$ subregions*

| | | Case 1 | | Case 2 | | | Case 1 | | Case 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{\hat{H}}{H}$ | Iter. | Cond. # | Iter. | Cond. # | $\frac{H}{h}$ | Iter. | Cond. # | Iter. | Cond. # |
| 3 | 9 | 2.6603 | 9 | 2.2559 | 3 | 9 | 2.6603 | 9 | 2.2559 |
| 4 | 9 | 3.0446 | 10 | 2.5183 | 4 | 9 | 2.7261 | 10 | 2.3299 |
| 5 | 10 | 3.3570 | 11 | 2.7782 | 5 | 10 | 2.8381 | 10 | 2.4353 |
| 6 | 10 | 3.6402 | 11 | 3.0078 | 6 | 10 | 2.9601 | 11 | 2.5488 |

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we use the first preconditioner $\widetilde{M}^{-1}$. We take the coefficient $\rho\equiv1$ in Case 1. In Case 2, $\rho$ is constant in one direction with a checkerboard pattern in the cross sections, where we take $\rho=1$ or $\rho=100$. The coefficients in both cases satisfy [10, Assumption 6.27.2], i.e., for all pairs of subdomains which have a vertex but not an edge in common, there exists an acceptable edge path (see [10, Definition 6.26]) between these two subdomains. Table 1 gives the iteration counts and condition number estimates with a change of the number of subregions. We find that the condition numbers are independent of the number of subregions. Table 2 gives results with a change of the number of subdomains and the size of the subdomain problems.

In the second set of experiments, we use the second preconditioner $\widehat{M}^{-1}$ and take the coefficient $\rho\equiv1$. We use the Preconditioned Conjugate Gradient (PCG) to estimate the largest eigenvalue of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)$, which is approximately 2.3249. For $18\times18\times18$ subdomains and $\frac{H}{h}=3$, we have a condition number estimate of 1.8767 for the two-level preconditioned BDDC operator. We select different values of $u$, the upper bound eigenvalue estimate of the preconditioned system, and $k$ to see how the condition number changes. We take $u=2.3$ and $u=3$ in Table 3 and 4, respectively. We also evaluate $C_1(k)$ for $k=1,2,3,4,5$. From these two tables, we find that the smallest eigenvalue is bounded from below by $C_1(k)$ and the condition number estimate becomes closer to 1.8767, the value for the two-level case, as k increases. We also see that if we can get more precise estimate for the largest eigenvalue of $\left(\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}^T\widetilde{T}^{-1}\widehat{\widetilde{R}}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{\widetilde{R}}_{\widehat{\Gamma}}^T\widetilde{T}\widehat{\widetilde{R}}_{\widehat{\Gamma}}\right)$, we need fewer Chebyshev iterations to

TABLE 3

*Eigenvalue bounds and iteration counts with the preconditioner $\widehat{M}^{-1}$, $u = 2.3$, $3 \times 3 \times 3$ subregions, $\frac{\hat{H}}{H} = 6$ and $\frac{H}{h} = 3$*

| k | Iter. | $C_1(k)$ | $\lambda_{min}$ | $\lambda_{max}$ | Cond. # |
|---|---|---|---|---|---|
| 1 | 13 | 0.6061 | 0.6167 | 2.3309 | 3.7797 |
| 2 | 9 | 0.9159 | 0.9255 | 1.8968 | 2.0496 |
| 3 | 8 | 0.9827 | 1.0000 | 1.8835 | 1.8836 |
| 4 | 8 | 0.9964 | 1.0016 | 1.8854 | 1.8825 |
| 5 | 8 | 0.9993 | 1.0009 | 1.8797 | 1.8780 |

TABLE 4

*Eigenvalue bounds and iteration counts with the preconditioner $\widehat{M}^{-1}$, $u = 3$, $3\times3\times3$ subregions, $\frac{\hat{H}}{H} = 6$ and $\frac{H}{h} = 3$*

| k | Iter. | $C_1(k)$ | $\lambda_{min}$ | $\lambda_{max}$ | Cond. # |
|---|---|---|---|---|---|
| 1 | 15 | 0.5000 | 0.5093 | 2.0150 | 3.9562 |
| 2 | 10 | 0.8571 | 0.8678 | 1.9744 | 2.2753 |
| 3 | 8 | 0.9615 | 0.9900 | 1.8821 | 1.9012 |
| 4 | 8 | 0.9897 | 1.0015 | 1.8955 | 1.8927 |
| 5 | 8 | 0.9972 | 1.0020 | 1.8903 | 1.8866 |

get a condition number, close to that of the two-level case. However, the iteration count is not very sensitive to the choice of $u$.

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