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A DOMAIN DECOMPOSITION DISCRETIZATION OF PARABOLIC PROBLEMS

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Abstract. In recent years, domain decomposition methods have attracted much attention due to their successful application to many elliptic and parabolic problems. Domain decomposition methods treat problems based on a domain substructuring, which is attractive for parallel computation, due to the independence among the subdomains. In principle, domain decomposition methods may be applied to the system resulting from a standard discretization of the parabolic problems or, directly, be carried out through a direct discretization of parabolic problems. In this paper, a direct domain decomposition method is introduced to discretize the parabolic problems. The stability and convergence of this algorithm are analyzed, and an $O(\tau + h)$ error bound is provided.

Key words. domain decomposition, parabolic problem, finite element

AMS subject classifications. 65F10, 65N30

1. Introduction. Domain decomposition methods are becoming popular algorithms for the numerical solutions of partial differential equations (PDEs) such as parabolic problems. Several strategies can be applied to obtain such algorithms. Among them, a first approach uses the standard discretization of parabolic problems (e.g., the backward Euler, Crank Nicolson), followed by applying domain decomposition methods to the resulting systems, as an iterative method as for elliptic problems (for references, see [1], [5] and literature therein). In contrast, a second approach is based on the discretization of the parabolic problems which leads to a domain decomposition algorithm as a direct method (for references, see [3], [6] and the literature therein). These strategies have proved, theoretically and practically, to be very effective for parallel computation.

In this paper, a domain decomposition method is introduced for parabolic problems based on the second approach. For a second order parabolic equation in $(0, T) \times \Omega$, where Ω is a polygonal region in two dimensional space, we consider an approximation of an initial-boundary value problem. This problem is directly discretized by a finite difference method with respect to the time variable t and by a finite element method with respect to the spatial variables $x = (x_1, x_2)$, leading to a direct domain decomposition method. Such a special discretization of the original problem results in a well-suited algorithm for parallel computation. The algorithm discussed can be viewed as a domain decomposition analog of the well known ADI methods for finite difference approximation of parabolic problems, see [2]. Here we prove that the resulting discrete problem approximates the original problem, and that this algorithm is stable and convergent with an error bound $O(\tau + h)$ in an appropriate norm. The error bound obtained for the method is the same as for the backward Euler scheme. To the best of our knowledge, this is the best error estimate known in the literature for this type of discretization.

The method discussed has previously been described in brief in [3]. A theorem formulated there (without proof) gives an error bound $O(\tau^{1/2} + h)$ provided that τ is

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proportional to h. Here, we improve our estimate by using refined tools.

The rest of the paper is organized as follows. In Section 2, the original problem and its special discretization are described. Section 3 is devoted to an analysis of the stability of the discrete problem. In Section 4, we prove that the method is convergent with an $O(\tau + h)$ error bound. Finally, some computational results are presented in Section 5.

2. The continuous and discrete problems. We consider a parabolic problem of the form: find $u \in L^2((0,T); H_0^1(\Omega)) \cap C^0((0,T); L^2(\Omega))$ such that

$$(\frac{\partial u}{\partial t}, v)_{L^2(\Omega)} + a(u,v) = (f,v)_{L^2(\Omega)}, \quad v \in H_0^1(\Omega), \quad \text{a.e. } t \in (0,T) \tag{2.1}$$

$$(u,v)_{L^2(\Omega)} = (u_0, v)_{L^2(\Omega)}, \quad v \in L^2(\Omega) \tag{2.2}$$

where

$$a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)}$$

and Ω is a polygonal region in R^2. We assume that $f \in L^2((0,T); L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$. The problem (2.1) has a unique solution and is stable, see for example [4].

We solve the problem (2.1) using a finite difference method and finite element method for the t and x variables, respectively. The interval $[0,T]$ is partitioned uniformly, $t_n = n\tau, n = 0, \cdots, N, N\tau = T$. A triangulation of Ω is constructed as follows: first Ω is divided into triangular or quadrilateral substructures Ω_i with a diameter H_i, where $i = 1, \cdots, I$; a coarse triangulation is formed with a parameter $H = max_i H_i$. We then divide each Ω_i into triangles $e_i^{(j)}$ with a diameter h_i. The resulting triangulation is conforming across $\partial\Omega_i$, and is called the fine triangulation with a parameter $h = max_i h_i$. We assume that the coarse and the fine triangulations are shape regular in the sense common to finite element theory. Let $V^h(\Omega)$ be the space of continuous, piecewise linear functions on the fine triangulation which vanish on $\partial\Omega$.

We will assume that there is a red-black ordering of the substructures Ω_i. Thus no two substructures of the same type share an edge. Let Ω^B and Ω^R denote the union of the black and red substructures, respectively. Let $\Gamma = \partial\Omega^R \backslash \partial\Omega$ and $\bar{\Omega}^B = \Omega^B \cup \Gamma$.

Define

$$(u,v)_{L_h^2(\Omega)} = \sum_{x_k \in \Omega_h} w_k u(x_k) v(x_k), \tag{2.3}$$

where Ω_h is the set of nodal points of Ω and the w_k are weights. With $\mathcal{N}(x_k)$ the union of the elements that share a nodal point x_k, we take $w_k = \sum_{i \in \mathcal{N}(x_k)} 1/6|detB_i|$, where B_i is the nonsingular matrix of the affine mapping between this element and the reference triangle.

The problem (2.1) is approximated by the following scheme: for $n = 0, \cdots, N-1$, find $U^{n+1} \in V^h(\Omega)$ such that, for $\forall v \in V^h(\Omega)$,

$$\begin{cases} (U_t^n, v)_{L_h^2(\Omega)} + a_R((U^{n+1/2} + U^n)/2, v) = (f_R^{n+1}, v)_{L^2(\Omega^R)}, \\ U^{n+1/2}(x) = U^n(x), \quad x \in \Omega_h^B, \end{cases} \tag{2.4}$$

$$\begin{cases} (U_t^{n+1/2}, v)_{L_h^2(\Omega)} + a_B((U^{n+1} + U^{n+1/2})/2, v) = (f_B^{n+1}, v)_{L^2(\Omega^B)}, \\ U^{n+1}(x) = U^{n+1/2}(x), \quad x \in \Omega_h^R, \end{cases} \tag{2.5}$$

and

(2.6) $$(U^0, v)_{L^2(\Omega)} = (u_0, v), \quad v \in V^h(\Omega).$$

Here

$$U_t^n \equiv (U^{n+1/2} - U^n)/\tau, \qquad U_t^{n+1/2} \equiv (U^{n+1} - U^{n+1/2})/\tau$$

and Ω_h^B and Ω_h^R are the sets of interior nodal points of Ω^B and Ω^R, respectively. We have

$$a(u,v) = a_R(u,v) + a_B(u,v),$$

where $a_R(\cdot,\cdot)$ and $a_B(\cdot,\cdot)$ are the restrictions of $a(\cdot,\cdot)$ to Ω^R and Ω^B, respectively, and

$$f^n(x) \equiv f(n\tau, x) = f_R^n(x) + f_B^n(x),$$

where $f_R^n(x) = 0$ at $x \in \Omega_h^B$ and $f_B^n(x) = 0$ at $x \in \Omega_h^R$.

Let us comment on the scheme (2.4)-(2.6). We note that (2.4) and (2.5) separately do not approximate the equation (2.1) in the standard sense. However, the sum of the equations (2.4) and (2.5) approximates the equation (2.1) with an error $O(\tau + h)$ for a sufficiently smooth solution u of (2.1). To see this, let us fix the level n, i.e. $t_n = n\tau$, and set $u^n(x) = u(n\tau, x)$. We have

$$a_R((u^{n+1/2} + u^n)/2, v) = a_R(u^n, v) + O(\tau),$$

and

$$a_B((u^{n+1} + u^{n+1/2})/2, v) = a_B(u^n, v) + O(\tau).$$

Hence $a_B(\cdot,\cdot) + a_R(\cdot,\cdot)$ approximates $a(\cdot,\cdot)$ with an error of $O(\tau)$. The right hand sides of (2.4)-(2.5) approximate $(f^{n+1}, v)_{L^2(\Omega)}$ also with an error $O(\tau)$ on the nth level. It is obvious that the approximation of $\partial u/\partial t$ by u_t^n and $u_t^{n+1/2}$ has an error $O(\tau)$ at interior nodal points of Ω_h^R and Ω_h^B. For points common to $\partial\Omega_i^R$ and $\partial\Omega_i^B$, the same follows from the fact that at those points

$$u_t^n + u_t^{n+1/2} = (u^{n+1} - u^n)/\tau.$$

In the following sections, we prove the stability and convergence of the scheme (2.4)-(2.6) in an appropriately chosen norm.

Let us briefly discuss the implementation of (2.4)-(2.6). Let $U^n(x)$ be given. We first need to find $U^{n+1/2}(x)$. The values of $U^{n+1/2}(x)$ at the interior nodal points of Ω^B are equal to $U^n(x)$ by definition, see (2.4). There remains to compute $U^{n+1/2}(x)$ at the nodal points of $\bar{\Omega}^R$ by solving (2.4). This reduces to solving a set of local Neumann problems on $\bar{\Omega}_i^R$ which are weakly coupled by equations for the unknowns at the substructure vertices of Ω_i^R. This system can be solved by block Gauss elimination reducing the system to a small system with the subdomain vertex unknowns only. Knowing the vertex unknowns, the system reduces to local independent problems on the Ω_i^B. Each local problem has a unique solution. Another approach to solving the systems which are weakly coupled at substructure vertices can be found in [5, Section 6.4] in connection with the FETI-DP algorithm. It should be pointed out that the coupling in the system acts as a coarse space, necessary for domain decomposition methods for elliptic discretizations to obtain an algorithm with a rate of convergence independent of the number of substructures. To compute $U^{n+1}(x)$, we solve (2.5) in the same way as (2.4) described above.

3. Stability. In this section, we prove the stability of (2.4)-(2.6) in an appropriate norm.

THEOREM 3.1. *The solution of (2.4)-(2.6) satisfies the following inequality:*

$$\max_{0\le n\le N} \| U^n \|^2_{L^2_h(\Omega)} + \tau \sum_{n=0}^{N-1} \{ |U^{n+1/2}+U^n|^2_{H^1(\Omega^R)} + |U^{n+1}+U^{n+1/2}|^2_{H^1(\Omega^B)} \} \tag{3.1}$$
$$\le M\{ \| U^0 \|^2_{L^2(\Omega)} + \tau \sum_{n=0}^{N-1} (\| f_R^{n+1} \|^2_{L^2(\Omega)} + \| f_B^{n+1} \|^2_{L^2(\Omega)}) \},$$

where M is a positive constant independent of h, H, and τ.

Proof. We take $v = 2\tau(U^{n+1/2}+U^n)$ and $v = 2\tau(U^{n+1}+U^{n+1/2})$ in (2.4) and (2.5), respectively. Adding the resulting equations, we obtain

$$\begin{aligned} &2\tau(U_t^n, U^{n+1/2}+U^n)_{L^2_h(\Omega)} + 2\tau(U_t^{n+1/2}, U^{n+1}+U^{n+1/2})_{L^2_h(\Omega)} \\ &+ \tau|U^{n+1/2}+U^n|^2_{H^1(\Omega^R)} + \tau|U^{n+1}+U^{n+1/2}|^2_{H^1(\Omega^B)} \\ &= 2\tau(f_R^{n+1}, U^{n+1/2}+U^n)_{L^2(\Omega^R)} + 2\tau(f_B^{n+1}, U^{n+1}+U^n)_{L^2(\Omega^B)}. \end{aligned} \tag{3.2}$$

We note that

$$\tau(U_t^n, U^{n+1/2}+U^n)_{L^2_h(\Omega)} = \| U^{n+1/2} \|^2_{L^2_h(\Omega)} - \| U^n \|^2_{L^2_h(\Omega)},$$

and

$$\tau(U_t^{n+1/2}, U^{n+1}+U^{n+1/2})_{L^2_h(\Omega)} = \| U^{n+1} \|^2_{L^2_h(\Omega)} - \| U^{n+1/2} \|^2_{L^2_h(\Omega)},$$

where we use that

$$\begin{aligned} &\tau(U_t^n, U^{n+1/2}+U^n)_{L^2_h(\Omega)} = \tau(U_t^n, U^{n+1/2}+U^n)_{L^2_h(\bar\Omega^R)} \\ &= \| U^{n+1/2} \|^2_{L^2_h(\bar\Omega^R)} - \| U^n \|^2_{L^2_h(\bar\Omega^R)} = \| U^{n+1/2} \|^2_{L^2_h(\Omega)} - \| U^n \|^2_{L^2_h(\Omega)} . \end{aligned}$$

Using these identities in (3.2) and summing the resulting equation with respect to n from 0 to k, we obtain

$$\begin{aligned} &2 \| U^{k+1} \|^2_{L^2_h(\Omega)} + \tau \sum_{n=0}^{k} \{ |U^{n+1/2}+U^n|^2_{H^1(\Omega^R)} + |U^{n+1}+U^{n+1/2}|^2_{H^1(\Omega^B)} \} \\ &= 2 \| U^0 \|^2_{L^2_h(\Omega)} + 2\tau \sum_{n=0}^{k} \{ (f_R^{n+1}, U^{n+1/2}+U^n)_{L^2(\Omega^R)} + (f_B^{n+1}, U^{n+1}+U^{n+1/2})_{L^2(\Omega^B)} \}. \end{aligned} \tag{3.3}$$

The second and third terms of the right hand side of (3.3) are estimated as follows. We have, for any $\varepsilon > 0$,

$$2(f_R^{n+1}, U^{n+1/2}+U^n)_{L^2(\Omega^R)} \le \frac{1}{\varepsilon} \| f_R^{n+1} \|^2_{L^2(\Omega)} + \varepsilon \| U^{n+1/2}+U^n \|^2_{L^2(\Omega)}, \tag{3.4}$$

and

$$2(f_B^{n+1}, U^{n+1}+U^{n+1/2})_{L^2(\Omega^B)} \le \frac{1}{\varepsilon} \| f_B^{n+1} \|^2_{L^2(\Omega)} + \varepsilon \| U^{n+1}+U^{n+1/2} \|^2_{L^2(\Omega)} . \tag{3.5}$$

We now estimate the second terms of the right hand sides of (3.4) and (3.5). We have

$$\| U^{n+1/2} + U^n \|^2_{L^2(\Omega)} \leq M(\| U^{n+1} \|^2_{L^2(\Omega)} + \| U^n \|^2_{L^2(\Omega)} + \sum_{i=1}^{I} h \| U^{n+1/2} + U^n \|^2_{L^2(\partial\Omega_i^R)}). \tag{3.6}$$

Note that here and below I reduces to the number of Ω_i^R substructures.

We show below that

$$\sum_{i=1}^{I} h \| U^{n+1/2} + U^n \|^2_{L^2(\partial\Omega_i^R)} \leq M|U^{n+1/2} + U^n|^2_{H^1(\Omega^R)}. \tag{3.7}$$

Using this in (3.6) and knowing that $\| v \|_{L_h^2(\Omega)}$ is equivalent to $\| v \|_{L^2(\Omega)}$ for $v \in V^h(\Omega)$, we obtain

$$\| U^{n+1/2} + U^n \|^2_{L^2(\Omega)} \leq M(\| U^{n+1} \|^2_{L_h^2(\Omega)} + \| U^n \|^2_{L_h^2(\Omega)} + |U^{n+1/2} + U^n|^2_{H^1(\Omega^R)}). \tag{3.8}$$

In a similar way, we show that

$$\| U^{n+1} + U^{n+1/2} \|^2_{L^2(\Omega)} \leq M\{\| U^{n+1} \|^2_{L_h^2(\Omega)} + \| U^n \|^2_{L_h^2(\Omega)} + |U^{n+1} + U^{n+1/2}|^2_{H^1(\Omega^B)}\}. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.4) and (3.5) and the resulting inequalities into (3.3), we obtain

$$(2 - 2M\varepsilon) \| U^{k+1} \|^2_{L_h^2(\Omega)} + \tau \sum_{n=0}^{k} (1 - M\varepsilon)\{|U^{n+1/2} + U^n|^2_{H^1(\Omega^R)} + |U^{n+1} + U^{n+1/2}|^2_{H^1(\Omega^B)}\}$$

$$\leq M\{\| U^0 \|^2_{L_h^2(\Omega)} + \tau \sum_{n=0}^{k} (\| U^n \|^2_{L_h^2(\Omega)} + \| f_R^{n+1} \|^2_{L^2(\Omega)} + \| f_B^{n+1} \|^2_{L^2(\Omega)})\}.$$

Choosing $\varepsilon \leq (2M)^{-1}$ and applying the Gronwall inequality, we obtain (3.1).

There remains to prove (3.7). Let $z \equiv U^{n+1/2} + U^n$ and let $I_H z$ be the linear and bilinear interpolant of z on triangular and quadrilateral substructures $\{\Omega_i\}$, respectively, by using values at the substructure vertices. We have

$$\sum_{i=1}^{I} h \| z \|^2_{L^2(\partial\Omega_i^R)} \leq 2 \sum_{i=1}^{I} h(\| z - I_H z \|^2_{L^2(\partial\Omega_i^R)} + \| I_H z \|^2_{L^2(\partial\Omega_i^R)}). \tag{3.10}$$

Using a simple trace theorem and a discrete Sobolev inequality, see [5, Remark 4.13], we obtain

$$\| z - I_H z \|^2_{L^2(\partial\Omega_i^R)} \leq M H_i (1 + \log \frac{H}{h}) |z|^2_{H^1(\Omega_i^R)}, \tag{3.11}$$

where H_i is the diameter of Ω_i^R. The second term of (3.10) is estimated as

$$\sum_{i=1}^{I} \| I_H z \|^2_{L^2(\partial\Omega_i^R)} \leq M \left(\sum_{i=1}^{I} H_i (|I_H z|^2_{H^1(\Omega_i^R)} + \frac{1}{H_i^2} \| I_H z \|^2_{L^2(\Omega_i^R)}) \right)$$

$$\leq M H^{-1} |I_H z|^2_{H^1(\Omega)} \leq \frac{M}{H} (1 + \log \frac{H}{h}) |z|^2_{H^1(\Omega^R)}. \tag{3.12}$$

We have used first the Friedrichs inequality on Ω, since $z = 0$ on $\partial\Omega$, and then the discrete Sobolev inequality. Substituting (3.11) and (3.12) into (3.10), we obtain

$$\sum_{i=1}^{I} h \parallel z \parallel^2_{L^2(\partial\Omega_i^R)} \leq M \frac{h}{H}(1+\log\frac{H}{h})|z|^2_{H^1(\Omega^R)}.$$

From this, (3.7) follows since $\frac{h}{H}(1+\log\frac{H}{h})$ is bounded.

In a similar way, we prove

$$\sum_{i=1}^{I} h \parallel U^{n+1} + U^{n+1/2} \parallel^2_{L^2(\partial\Omega_i^B)} \leq M \frac{h}{H}(1+\log\frac{H}{h})|U^{n+1} + U^{n+1/2}|^2_{H^1(\Omega^B)}.$$

This is needed to prove (3.9). ☐

4. Convergence. In this section, we prove the convergence of the discrete problem (2.4)-(2.6) to the original problem (2.1). We first prove the following two lemmas.

LEMMA 4.1. *Let Ω_i^N be a subdomain Ω_i^R or Ω_i^B, let Ω_i^N be convex, and let v be harmonic on Ω_i^N. Then,*

$$\parallel v \parallel_{H^{-1/2}(\partial\Omega_i^N)} \leq C \parallel v \parallel_{L^2(\Omega_i^N)}, \tag{4.1}$$

where C independent of h and H_i.

Proof. We first prove (4.1) for $D = \Omega_i^N$ with a diameter on the order of one. By definition

$$\parallel v \parallel_{H^{-1/2}(\partial D)} = \sup_{g\in H^{1/2}(\partial D)} \frac{(v,g)_{L^2(\partial D)}}{\parallel g \parallel_{H^{1/2}(\partial D)}}. \tag{4.2}$$

Let $w \in H^2(D) \cap H_0^1(D)$ be the solution of the problem:

$$\Delta^2 w = 0 \text{ in } D, \ \ w = 0 \text{ and } \frac{\partial w}{\partial n} = g \text{ on } \partial D.$$

It is known that

$$\parallel w \parallel_{H^2(D)} \leq M \parallel g \parallel_{H^{1/2}(\partial D)} .$$

In addition,

$$\int_D (-\Delta w) v dx = \int_D \nabla w \nabla v dx + \int_{\partial D} \frac{\partial w}{\partial n} v ds = \int_{\partial D} g v ds,$$

since v is harmonic. Thus,

$$|\int_{\partial D} g v ds| \leq \parallel w \parallel_{H^2(D)} \parallel v \parallel_{L^2(D)} \leq M \parallel g \parallel_{H^{1/2}(\partial D)} \parallel v \parallel_{L^2(D)} .$$

Using this in (4.2), we obtain

$$\parallel v \parallel_{H^{-1/2}(\partial D)} \leq M \parallel v \parallel_{L^2(D)} .$$

This is also valid for an Ω_i^N of diameter H_i by a scaling argument. ☐

Lemma 4.2. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v \in V^h(\Omega)$. Given any $\varepsilon > 0$, there exists a constant M, independent of h, such that*

$$|(u,v)_{L^2(\Omega)} - (u,v)_{L_h^2(\Omega)}| \le Mh^2 \parallel u \parallel_{H^2(\Omega)}^2 + \varepsilon \parallel v \parallel_{L_h^2(\Omega)}^2 . \tag{4.3}$$

Proof. Let $\hat{u}$ be the piecewise linear interpolant of u in the finite element space $V^h(\Omega)$. We have

$$|(u,v)_{L^2(\Omega)} - (u,v)_{L_h^2(\Omega)}| \le |(u-\hat{u},v)_{L^2(\Omega)}| + |(\hat{u},v)_{L^2(\Omega)} - (u,v)_{L_h^2(\Omega)}|. \tag{4.4}$$

The first term in the right hand side of (4.4) can be estimated by an interpolation theorem:

$$|(u-\hat{u},v)_{L^2(\Omega)}| \le M\|u-\hat{u}\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le Mh^2\|u\|_{H^2(\Omega)}\|v\|_{L_h^2(\Omega)}, \tag{4.5}$$

where we use that $\|v\|_{L^2(\Omega)}$ is equivalent to $\|v\|_{L_h^2(\Omega)}$ for $v \in V^h(\Omega)$.

We now prove an estimate for the second term in the right hand side of (4.4). Let $u_l = u(x_l)$. Since, $u \in H^2(\Omega)$ and $\hat{u}$ is the piecewise linear interpolant, we have:

$$\sum_{x_i \in \Omega_h, x_j \in \mathcal{N}(x_i)} (u_i - u_j)^2 = \sum_{x_i \in \Omega_h, x_j \in \mathcal{N}(x_i)} (\hat{u}_i - \hat{u}_j)^2 \le C|\hat{u}|_{H^1(\Omega)}^2. \tag{4.6}$$

Let us consider an element e. Denote by B the nonsingular matrix of the affine mapping between e and the reference triangle. The mass matrix of this triangle is

$$\mathcal{M} = |detB| \begin{pmatrix} 1/12 & 1/24 & 1/24 \\ 1/24 & 1/12 & 1/24 \\ 1/24 & 1/24 & 1/12 \end{pmatrix}.$$

Therefore,

$$(\hat{u},v)_{L^2(e)} = (u_1,u_2,u_3)\mathcal{M}(v_1,v_2,v_3)^T = |detB|(1/12 \sum_{i=1,2,3} u_i v_i + 1/24 \sum_{i,j=1,2,3; i\neq j} u_i v_j).$$

We sum over all the elements and remark that $(u,v)_{L_h^2(\Omega)} = (\hat{u},v)_{L_h^2(\Omega)}$. By (2.3) and (4.6), we obtain:

$$\begin{aligned} & |(\hat{u},v)_{L^2(\Omega)} - (\hat{u},v)_{L_h^2(\Omega)}| \\ & \le Mh^2 \sum_{x_i \in \Omega_h, x_j \in \mathcal{N}(x_i)} (u_i - u_j) v_i \\ & \le Mh^2 \sqrt{\left(\sum_{x_i \in \Omega_h, x_j \in \mathcal{N}(x_i)} (u_i - u_j)^2 \right) \left(\sum_{x_i \in \Omega_h} v_i^2 \right)} \\ & \le Mh|\hat{u}|_{H^1(\Omega)}\|v\|_{L_h^2(\Omega)} \\ & \le Mh\|u\|_{H^2(\Omega)}\|v\|_{L_h^2(\Omega)}. \end{aligned} \tag{4.7}$$

Substituting (4.5) and (4.7) into (4.4)and using the ε inequality, we obtain (4.3). ∎

THEOREM 4.3. *Let $(\partial u/\partial t) \in L^2((0,T), H^2(\Omega))$ and $(\partial f/\partial t) \in L^2((0,T), L^2(\Omega))$. Then there exists a constant M, independent of h, H, and τ, such that*

$$(4.8)\quad \begin{aligned} &\max_{0\le n\le N} \| z^n \|^2_{L^2_h(\Omega)} + \tau \left(\sum_{n=0}^{N-1} \{ |z^{n+1/2} + z^n|^2_{H^1(\Omega^R)} \right. \\ &\left. + |z^{n+1} + z^{n+1/2}|^2_{H^1(\Omega^B)} \} \right) \le M(\tau^2 + h^2), \end{aligned}$$

where $z^n = u(n\tau, x) - U^n(x)$, $z^{n+1/2} = u((n+1)\tau, x) - U^{n+1/2}(x)$, and u and $(U^{n+1/2}, U^{n+1})$ are the solutions of (2.1) and (2.4)-(2.6), respectively.

Proof. Let

$$v^n = U^n - W^n, \quad v^{n+1} = U^{n+1} - W^{n+1}, \quad v^{n+1/2} = U^{n+1/2} - W^{n+1},$$

where W^n is an interpolant of $u(n\tau, x)$ in the finite element space $V^h(\Omega)$. Let

$$\tilde{z}^n = u(n\tau, x) - W^n(x), \quad W^n_{\tilde{t}} = (W^{n+1} - W^n)/\tau.$$

We first substitute $U^n = v^n + W^n, U^{n+1} = v^{n+1} + W^{n+1}$ and $U^{n+1/2} = v^{n+1/2} + W^{n+1}$ into (2.4) and (2.5). In the resulting equations, we then set $v = 2\tau(v^{n+1/2} + v^n)$ and $v = 2\tau(v^{n+1} + v^{n+1/2})$, respectively, as in the proof of Theorem 3.1. The equations obtained are of the form

$$(4.9)\quad \left\{ \begin{array}{l} 2\tau(v^n_t, v^{n+1/2} + v^n)_{L^2_h(\Omega)} + \tau |v^{n+1/2} + v^n|^2_{H^1(\Omega^R)} = \\ 2\tau\{(f^{n+1}_R, v^{n+1/2} + v^n)_{L^2(\Omega^R)} - 2(W^n_{\tilde{t}}, v^{n+1/2} + v^n)_{L^2_h(\Omega)}\} \\ -\tau(\nabla(W^{n+1} + W^n), \nabla(v^{n+1/2} + v^n))_{L^2(\Omega^R)}, \\ \\ v^{n+1/2} = v^n - \tau W^n_{\tilde{t}}, \quad x \in \Omega^B_h, \end{array} \right.$$

and

$$(4.10)\quad \left\{ \begin{array}{l} 2\tau(v^{n+1/2}_t, v^{n+1} + v^{n+1/2})_{L^2_h(\Omega)} + \tau |v^{n+1} + v^{n+1/2}|^2_{H^1(\Omega^B)} = \\ 2\tau\{(f^{n+1}_B, v^{n+1} + v^{n+1/2})_{L^2(\Omega^B)} \\ -(\nabla W^{n+1}, \nabla(v^{n+1} + v^{n+1/2}))_{L^2(\Omega^B)}\}, \\ \\ v^{n+1} = v^{n+1/2}, \quad x \in \Omega^R_h. \end{array} \right.$$

Adding these equations, we have, cf. (3.3),

$$(4.11)\quad \begin{aligned} &2(\| v^{n+1} \|^2_{L^2_h(\Omega)} - \| v^n \|^2_{L^2_h(\Omega)}) + \tau\{|v^{n+1/2} + v^n|^2_{H^1(\Omega^R)} + \\ &|v^{n+1} + v^{n+1/2}|^2_{H^1(\Omega^B)}\} = \tau\{2(f^{n+1}_R, v^{n+1/2} + v^n)_{L^2(\Omega^R)} + \\ &2(f^{n+1}_B, v^{n+1} + v^{n+1/2})_{L^2(\Omega^B)} - (\nabla(W^{n+1} + W^n), \nabla(v^{n+1/2} + v^n))_{L^2(\Omega^R)} \\ &-2(W^n_{\tilde{t}}, v^{n+1/2} + v^n)_{L^2_h(\Omega)} - 2(\nabla W^{n+1}, \nabla(v^{n+1} + v^{n+1/2})_{L^2(\Omega^B)}\}. \end{aligned}$$

We note that for $u^n(x) \equiv u(n\tau, x)$, where $u(t, x)$ is the solution of (2.1), it holds that

$$(4.12)\quad \begin{aligned} &2(u^n_{\tilde{t}}, v^{n+1/2} + v^n)_{L^2(\Omega)} + (\nabla(u^{n+1} + u^n), \nabla(v^{n+1/2} + v^n))_{L^2(\Omega)} \\ &= 2(f^{n+1}, v^{n+1/2} + v^n)_{L^2(\Omega)} + (\varrho(\tau), v^{n+1/2} + v^n)_{L^2(\Omega)}, \end{aligned}$$

with $\varrho(\tau) = O(\tau)$. Let the left hand side of (4.11) be denoted by $J(v^{n+1}, v^n)$. Using (4.12) in (4.11) and doing simple manipulations, we obtain

$$\begin{aligned}
J(v^{n+1}, v^n) &= \tau\{2\tau(f_B^{n+1}, v_{\bar{t}}^n)_{L^2(\Omega^B)} + 2(\tilde{z}_{\bar{t}}^n, v^{n+1/2} + v^n)_{L_h^2(\Omega)} \\
&+(\nabla(\tilde{z}^{n+1} + \tilde{z}^n), \nabla(v^{n+1/2} + v^n))_{L^2(\Omega^R)} \\
&+2(\nabla\tilde{z}^{n+1}, \nabla(v^{n+1} + v^{n+1/2}))_{L^2(\Omega^B)} - \tau(\nabla u_{\bar{t}}^n, \nabla(v^{n+1} + v^{n+1/2}))_{L^2(\Omega^B)} \\
&-\tau(\nabla(u^{n+1} + u^n), \nabla v_{\bar{t}}^n)_{L^2(\Omega^B)} - (\varrho(\tau), v^{n+1/2} + v^n)_{L^2(\Omega)} \\
&+(2(u_{\bar{t}}^n, v^{n+1/2} + v^n)_{L^2(\Omega)} - 2(u_{\bar{t}}^n, v^{n+1/2} + v^n)_{L_h^2(\Omega)})\}.
\end{aligned} \tag{4.13}$$

We now estimate each term of the right hand side of (4.13). We obtain

$$\begin{aligned}
&2\tau^2 \sum_{n=0}^{k-1} (f_B^{n+1}, v_{\bar{t}}^n)_{L^2(\Omega^B)} \\
&= 2\tau\{(f_B^k, v^k)_{L^2(\Omega^B)} - (f_B^0, v^0)_{L^2(\Omega^B)}\} - 2\tau^2 \sum_{n=0}^{k-1} (f_{B,\tilde{t}}^n, v^n)_{L^2(\Omega^B)}\} \\
&\leq \varepsilon \parallel v^k \parallel_{L^2(\Omega)}^2 + M\{\tau^2 \parallel f_B^k \parallel_{L^2(\Omega)}^2 + \tau^2 \parallel f_B^0 \parallel_{L^2(\Omega)}^2 + \parallel v^0 \parallel_{L^2(\Omega)}^2 \\
&\quad + \tau \sum_{n=0}^{k-1} (\tau^2 \parallel f_{B\tilde{t}}^n \parallel_{L^2(\Omega)}^2 + \parallel v^n \parallel_{L^2(\Omega)}^2)\}.
\end{aligned}$$

Using a Sobolev inequality, we have

$$\begin{aligned}
&2\tau^2 \sum_{n=0}^{k-1} (f_B^{n+1}, v_{\bar{t}}^n)_{L^2(\Omega)} \leq \varepsilon \parallel v^k \parallel_{L^2(\Omega)}^2 + M\{\parallel v^0 \parallel_{L^2(\Omega)}^2 + \\
&+\tau \sum_{n=0}^{k-1} \parallel v^n \parallel_{L^2(\Omega)}^2 + \tau \sum_{n=0}^{N-1} \tau^2(\parallel f_{B,\tilde{t}}^n \parallel_{L^2(\Omega)}^2 + \parallel f_B^n \parallel_{L^2(\Omega)}^2)\}.
\end{aligned} \tag{4.14}$$

The second term of (4.13) is estimated as

$$2(\tilde{z}_{\bar{t}}^n, v^{n+1/2} + v^n)_{L_h^2(\Omega)} \leq \varepsilon \parallel v^{n+1/2} + v^n \parallel_{L_h^2(\Omega)}^2 + \frac{1}{\varepsilon} \parallel \tilde{z}_t^n \parallel_{L_h^2(\Omega)}^2 .$$

Using that $\|v^{n+1/2} + v^n\|_{L^2(\Omega)}$ is equivalent to $\|v^{n+1/2} + v^n\|_{L_h^2(\Omega)}$, we know that the first term of the right hand side of this expression has already been estimated, see (3.8), while the second one is estimated by an interpolation theorem. Considering these terms, we obtain

$$\begin{aligned}
&2\tau(\tilde{z}_t^n, v^{n+1/2} + v^n)_{L_h^2(\Omega)} \leq M\varepsilon\tau\{\parallel v^{n+1} \parallel_{L_h^2(\Omega)}^2 \\
&+ \parallel v^n \parallel_{L_h^2(\Omega)}^2 + |v^{n+1/2} + v^n|_{H^1(\Omega^R)}^2\} + M\tau h^4 |u_{\bar{t}}^n|_{H^2(\Omega)}^2.
\end{aligned} \tag{4.15}$$

The third and fourth terms of (4.13) are estimated as

$$\begin{aligned}
&\tau(\nabla(\tilde{z}^{n+1} + \tilde{z}^n), \nabla(v^{n+1/2} + v^n))_{L^2(\Omega^R)} \\
&\leq \tau\varepsilon \parallel \nabla(v^{n+1/2} + v^n) \parallel_{L^2(\Omega^R)}^2 + M\tau h^2 |u^n + u^{n+1}|_{H^2(\Omega^R)}^2
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&2\tau(\nabla\tilde{z}^{n+1}, \nabla(v^{n+1} + v^{n+1/2}))_{L^2(\Omega^B)} \\
&\leq \tau\varepsilon \parallel \nabla(v^{n+1} + v^{n+1/2}) \parallel_{L^2(\Omega^B)}^2 + M\tau h^2 |u^{n+1}|_{H^1(\Omega^B)}^2.
\end{aligned} \tag{4.17}$$

The fifth term of (4.13) is estimated as

$$\begin{aligned}&\tau^2(\nabla u^n_{\tilde t}, \nabla(v^{n+1}+v^{n+1/2}))_{L^2(\Omega^B)}\\ &\le \frac{\tau^3}{4\varepsilon} \parallel \nabla u^n_{\tilde t} \parallel^2_{L^2(\Omega)} + \varepsilon\tau \parallel \nabla(v^{n+1}+v^{n+1/2}) \parallel^2_{L^2(\Omega^B)} .\end{aligned} \tag{4.18}$$

The sixth term of (4.13) is estimated as follows. Setting $g^{n+1} = u^{n+1} + u^n$, we have

$$\begin{aligned}-\tau^2\sum_{n=0}^{k-1}(\nabla g^{n+1}, \nabla v^n_{\tilde t})_{L^2(\Omega^B)} &= \tau(\nabla g^k, \nabla v^k)_{L^2(\Omega^B)}\\ &-\tau(\nabla g^0, \nabla v^0)_{L^2(\Omega^B)} + \tau^2\sum_{n=0}^{k-1}(\nabla g^n_{\tilde t}, \nabla v^n)_{L^2(\Omega^B)}.\end{aligned} \tag{4.19}$$

We have

$$\tau(\nabla g^k, \nabla v^k)_{L^2(\Omega^B)} = -\tau(\Delta g^k, v^k)_{L^2(\Omega^B)} + \sum_{i=1}^{I}\tau(\frac{\partial}{\partial n}g^k, v^k)_{L^2(\partial\Omega_i^B)}. \tag{4.20}$$

Using Lemma 4.1, we have

$$\begin{aligned}\tau(\frac{\partial}{\partial n}\nabla g^k, v^k)_{L^2(\partial\Omega_i^B)} &\le \tau|\frac{\partial}{\partial n}g^k|_{H^{1/2}(\partial\Omega_i^B)}|v^k|_{H^{-1/2}(\partial\Omega_i^B)}\\ &\le M\tau^2|g^k|^2_{H^2(\Omega_i^B)} + \varepsilon|v^k|^2_{L^2(\Omega_i^B)}.\end{aligned}$$

We have also used that v^k can be represented as $v^k = P_i v^k + \mathcal{H}_i v^k$ where $\mathcal{H}_i v^k$ is discrete harmonic on Ω_i^B with the value v^k on $\partial\Omega_i^B$ and P_i is the $H_0^1(\Omega_i)$ projection. Using this estimate in (4.20), we obtain

$$\tau(\nabla g^k, \nabla v^k)_{L^2(\Omega^B)} \le M\tau^2|g^k|^2_{H^2(\Omega^B)} + \varepsilon \parallel v^k \parallel^2_{L^2(\Omega^B)} . \tag{4.21}$$

The second and third terms of right hand side of (4.19) are estimated in the same way. Using these estimates in (4.19), we obtain

$$\begin{aligned}\tau^2\sum_{n=0}^{k-1}(\nabla g^{n+1}, \nabla v^n_{\tilde t})_{L^2(\Omega)} \le \quad &\varepsilon(\parallel v^k \parallel^2_{L^2(\Omega^B)} + \parallel v^0 \parallel^2_{L^2(\Omega^B)})\\ &+ M\{\tau\sum_{n=0}^{k-1}(\tau^2|g^n_{\tilde t}|^2_{H^2(\Omega^B)} + \parallel v^n \parallel^2_{L^2(\Omega^B)})\}.\end{aligned} \tag{4.22}$$

The seventh term of (4.13) is estimated using (3.8) and (4.12). Recall that $\varrho(\tau)$ is defined in (4.12). We have

$$\begin{aligned}\tau(\varrho(\tau), v^{n+1/2}+v^n)_{L^2(\Omega)} \le M\tau\varepsilon\{&\parallel v^{n+1} \parallel^2_{L^2_h(\Omega)} + \parallel v^n \parallel^2_{L^2_h(\Omega)}\\ &+|v^{n+1/2}+v^n|^2_{H^1(\Omega^R)}\} + M\tau^3 \parallel u^n_{\tilde t} \parallel^2_{H^2(\Omega)} .\end{aligned} \tag{4.23}$$

We estimate the last term of (4.13) by using Lemma 4.2. From Lemma 4.2, we have

$$\begin{aligned}&|(2(u^n_{\tilde t}, v^{n+1/2}+v^n)_{L^2(\Omega)} - 2(u^n_{\tilde t}, v^{n+1/2}+v^n)_{L^2_h(\Omega)})|\\ &\le Mh^2 \parallel u^n_{\tilde t} \parallel^2_{H^2(\Omega)} + \varepsilon \parallel v^{n+1/2}+v^n \parallel^2_{L^2_h(\Omega)} .\end{aligned} \tag{4.24}$$

TABLE 5.1
$u = e^t \times sin(\pi x) \times sin(\pi y)$

k	I	h	L_2 error	L_2^k/L_2^{k+1}
1	4	1/8	8.9594×10^{-2}	—
2	16	1/16	3.7832×10^{-2}	2.3682
3	64	1/32	1.4373×10^{-2}	2.6321
4	256	1/64	5.2540×10^{-3}	2.7356

Using that $\|v\|_{L^2(\Omega)}$ is equivalent to $\|v\|_{L_h^2(\Omega)}$ for $v \in V_h(\Omega)$ and (3.8), we obtain

$$\begin{aligned}(4.25)\qquad &\| v^{n+1/2} + v^n \|^2_{L_h^2(\Omega)}\\ \le& M\left(\| v^{n+1} \|^2_{L_h^2(\Omega)} + \| v^n \|^2_{L_h^2(\Omega)} + |v^{n+1/2} + v^n|^2_{H^1(\Omega^R)}\right).\end{aligned}$$

Therefore, the last term can be estimated as

$$\begin{aligned}(4.26)\quad &\tau|(2(u_{\tilde{t}}^n, v^{n+1/2} + v^n)_{L^2(\Omega)} - 2(u_{\tilde{t}}^n, v^{n+1/2} + v^n)_{L_h^2(\Omega)})|\\ &\le M\tau\left(h^2 \| u_{\tilde{t}}^n \|^2_{H^2(\Omega)} + \varepsilon(\| v^{n+1} \|^2_{L_h^2(\Omega)} + \| v^n \|^2_{L_h^2(\Omega)} + |v^{n+1/2} + v^n|^2_{H^1(\Omega^R)})\right).\end{aligned}$$

We now sum (4.13) over n from 0 to $k-1$ and then use the estimates above. This gives

$$\begin{aligned}&(2 - 3\varepsilon - 2M\varepsilon\tau) \| v^k \|^2_{L_h^2(\Omega)} + (1 - 3M\varepsilon)\tau \sum_{n=0}^{k-1}(|v^{n+1/2} + v^n|^2_{H^1(\Omega^R)}\\ &+ |v^{n+1} + v^{n+1/2}|^2_{H^1(\Omega^B)}) \le M\{\| v^0 \|^2_{L_h^2(\Omega)}\\ (4.27)\quad &+ \tau \sum_{n=0}^{N-1} \tau^2(\| f_B^{n+1} \|^2_{L^2(\Omega)} + \| f_{B\tilde{t}}^n \|^2_{L^2(\Omega)}) + \tau \sum_{n=0}^{k-1} \| v^n \|^2_{L_h^2(\Omega)}\\ &+ \tau \sum_{n=0}^{k-1}(\tau^2 + h^2)(\| u_{\tilde{t}}^n \|^2_{H^2(\Omega)} + \| u^{n+1} \|^2_{H^2(\Omega)} + \| u^n \|^2_{H^2(\Omega)})\}.\end{aligned}$$

Choosing a sufficiently small $\varepsilon > 0$ and using the Gronwall inequality, and the assumption on $u(t,x)$ and $f(t,x)$, we obtain

$$\| v^k \|^2_{L^2(\Omega)} + \tau \sum_{n=0}^{k-1}\{|v^{n+1/2} + v^n|^2_{H^1(\Omega^R)} + |v^{n+1} + v^{n+1/2}|^2_{H^1(\Omega^B)}\} \le M(\tau^2 + h^2).$$

From this (4.8) follows by using the triangle inequality and an interpolation theorem. ◻

5. Computational results. We have applied the algorithm to the model problem (2.1) and (2.2), where we take $\Omega = [0,1]^2$ and the time interval is $[0, 0.1]$, i.e., $T = 0.1$. We decompose the unit square into $I^{1/2} \times I^{1/2}$ subdomains with the sidelength $H = 1/I^{1/2}$ and the time interval into N subintervals with the length $\tau = \Delta t = 0.1/N$. (2.1) and (2.2), in each subdomain, is discretized by conforming piecewise linear elements with a mesh diameter h. We have carried out two different sets of experiments to test the convergence of the algorithm. In the first, we take the right hand side function

TABLE 5.2
$u = t \times x \times (1 - x) \times y \times (1 - y)$

k	I	h	L_2 error	L_2^k/L_2^{k+1}
1	4	1/8	5.9089×10^{-4}	—
2	16	1/16	2.2110×10^{-4}	2.6724
3	64	1/32	7.9217×10^{-5}	2.7911
4	256	1/64	2.8229×10^{-5}	2.8062

TABLE 5.3
$u = e^t \times sin(\pi x) \times sin(\pi y)$

k	I	h	L_2 error	L_2^k/L_2^{k+1}
1	8	1/8	4.4601×10^{-2}	—
2	16	1/16	3.7832×10^{-2}	1.1789
3	32	1/32	2.8777×10^{-2}	1.3147
4	64	1/64	2.1034×10^{-2}	1.3681
5	128	1/128	1.5110×10^{-2}	1.3920

$f = e^t \times sin(\pi x_1) \times sin(\pi x_2) \times (1+2\pi^2)$ and the initial data $u_0 = sin(\pi x_1) \times sin(\pi x_2)$. The exact solution to (2.1) and (2.2) is $u = e^t \times sin(\pi x_1) \times sin(\pi x_2)$. Fixing $I^{1/2} = 4$, we have 16 subdomains. We increased N and decreased h such that Δt is proportional to h^2. The results are given in Table 5.1. In a second set of the experiments, we chose $f = x_1 \times (1 - x_1) \times x_2 \times (1 - x_2) + 2 \times t \times x_2 \times (1 - x_2) + 2 \times t \times x_1 \times (1 - x_1)$ and $u_0 = 0$. The exact solution is $u = t \times x_1 \times (1 - x_1) \times x_2 \times (1 - x_2)$. We varied Δt and h as in the first set. The results are given in Table 5.2. These experiments show that our algorithm is stable and has a rate of convergence of $O(\tau + h)$. We note that that we have to choose Δt proportional to h^2 in order to get this rate of convergence. We also give the results by choosing Δt proportional to h in Table 5.3 and 5.4, respectively. Comparing the results in Tables 5.3 and 5.4 with those in Tables 5.1 and 5.2, respectively, reveals that the rates of convergence in Tables 5.3 and 5.4 are slower than those in 5.3 and 5.4. Nevertheless, convergence can be obtained in both cases. Limited by our computer, we cannot treat larger problems to test the rate of convergence when we take Δt proportional to h. It seems that we will obtain the rate of convergence at least $\sqrt{2}$.

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TABLE 5.4
$u = t \times x \times (1 - x) \times y \times (1 - y)$

k	I	h	L_2 error	L_2^k/L_2^{k+1}
1	8	1/8	2.7973×10^{-4}	—
2	16	1/16	2.2110×10^{-4}	1.2651
3	32	1/32	1.6043×10^{-4}	1.3782
4	64	1/64	1.1407×10^{-4}	1.4046
5	128	1/128	8.0728×10^{-5}	1.4130

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