The Two-Batch Liar Game over an Arbitrary Channel

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Abstract

We consider liar games in which player Paul must ask one full batch of questions, receive all answers, and then ask a second and final batch of questions. We show that the effect of this restriction is asymptotically negligible. The strategy for Paul is given explicitly.

1 Introduction

In this paper, we present a variant of the Rényi-Ulam game which is similar to the one we considered in [5]. The main difference consists in the type of strategy Paul is allowed to employ; the case we study here has Paul using a semi-offline strategy, as opposed to the completely online one of [5]. We introduce the game and present a few variants and recent results, comparing and contrasting them to the results in our current work.

1.1 History and recent results

In the original 2-player Rényi-Ulam game, Carole (player 1) thinks of a number $x \in \{1, \ldots, n\}$, while Paul (player 2) must find it by asking q Yes/No questions. The catch is that Carole is allowed to lie, but only at most k times (k being a fixed integer). The question is "for which n, q, k can Paul guess the number and win?"

Many researchers have examined this game and variants thereof; there is an extensive literature on the subject, of which we mention Pelc's excellent survey article [6]. For the reader interested in the history of the subject, good references are provided in Rényi [7], Ulam [10], and Berlekamp [2].

Historically, the full-lie version (where Carole is allowed to lie in whichever way she chooses, when she chooses to do it) was considered first; it is known that for this case, Carole can win when

$$2^q < n \left(\sum_{i=0}^{k} \binom{q}{i} \right) ,$$

and the converse is roughly true when n and q are large (while k is fixed; see [8]).

The more recently introduced half-lie case restricts Carole's ability to lie by requiring her to tell the truth when the truthful answer is Yes. Cicalese and Mundici [3] have shown that

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in the one half-lie case ($k = 1$), the maximal n for which Paul can win with q questions is asymptotic to $\frac{2^{q+1}}{q}$, as q goes to infinity.

In our paper [4], we have shown that for any fixed number k of half-lies, the asymptotics for the maximal value of n as a function of q and k as q goes to infinity are given by $\frac{2^{q+k}}{\binom{q}{k}}$.

There is a simple connection between the half-lie case and the Z-channel of Coding Theory, where during communication, a 0 can be accidentally transformed into a 1, but a 1 has to be always transmitted as a 1 (we allow for false positives, but not for false negatives; see Figure 1).

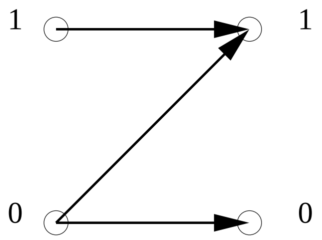

Figure 1: The Z-channel

Following this idea, we have further extended our results in [5], where we found asymptotics for arbitrary channels C.

Below we give a few definitions (which will be referred to throughout this paper) and state the main theorem from [5].

Definition 1. *A t-ary channel C is a set of ordered pairs (x, y) with $1 \leq x, y \leq t$, both integers, such that for each $1 \leq x \leq t$, $(x, x) \in C$. The pairs $(x, y) \in C$ with $y \neq x$ are called potential errors. The total number of potential errors is denoted by E.*

We include below a picture of an "arbitrary" channel (Figure 2) which we have also used in [5].

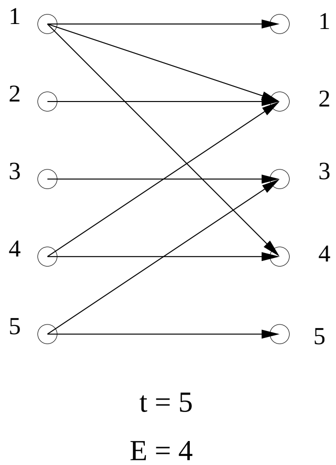

Figure 2: A 5-ary channel C, with $E = 4$

We now define (from [5]) the focus of our work, the (n, k, C)-liargame with q questions. There are two players, Paul and Carole, and q rounds. There is a set Ω of size n, the possibilities. Carole thinks of an $\alpha \in \Omega$. On each round Paul partitions Ω into disjoint sets $A_1, \dots, A_t$; Carole finds that i for which $\alpha \in A_i$ and responds with either i or some $j \neq i$ with $(i, j) \in C$. The latter case is called a lie or an error, we use the terms interchangably in this work. (We note that in the Coding Theory literature one wishes to send information through a channel and one wishes that it be deciphered accuratedly even when there are a certain number E or errors. These are called E-error correcting codes.)

Paul's choice of partitions in later rounds can, and in general will, depend on Carole's responses — hence we say that the channel allows for feedback.

Carole can make at most k lies in the course of the game. At the end of the q rounds Paul has won if and only if there is only one possible $\alpha \in \Omega$ for which Carole could have made her responses.

Definition 2. *Let $A_{C,k}(q)$ be the maximal n such that there is a winning strategy for Paul in the (n, k, C) game with q questions.*

Theorem 1.1. *(main result from [5]) Let C be an arbitrary (fixed) t-ary channel with $E > 0$ potential errors. Then for any fixed $k \in \mathbb{N}$,*

$$A_{C,k}(q) \sim \left(\frac{t}{E}\right)^k \frac{t^q}{\binom{q}{k}} ,$$

where the asymptotics are taken as $q \to \infty$.

We recall the vector format of [5].

Definition 3. *Consider an intermediate position in the game. For $0 \leq i \leq k$ let x_i be the number of possibilities for which Carole has lied precisely i times. We call the vector $\mathbf{x} = (x_0, \dots, x_k)$ the* state vector.

We remark that the state vector completely characterizes the state of the game, up to a renaming of Carole's possible choices. We further allow a game to have starting state $\mathbf{x} = (x_0, \dots, x_k)$. In such a game Carole has x_j possibilities for which she may lie $(k-j)$ times, $0 \leq j \leq k$.

1.2 Two batch strategies and the main result

In the Coding Theory setting described in the previous section, we have the following problem: Bob sends $x \in \{1, \dots, t\}^q$ to Alice through channel C, and the channel may make as many as k mistakes. Bob's full message is one of n possibilities. Is there a protocol by which correct reception of the message by Alice is assured?

The answer is yes if and only if Paul can win the (n, k, C) game with q questions *by asking all the questions in advance*. This is an additional and very strong constraint imposed on the game we have described in the previous section, which assumes complete feedback (each question is followed by an answer and each question is based on all previous answers, in a completely online strategy). The offline, no feedback constraint appears to change the problem drastically. We denote by $A^-_{C,k}(q)$ the maximal n for which Paul wins under the offline constraint. Clearly $A^-_{C,k}(q) \leq A_{C,k}(q)$. However, the asymptotics of $A^-_{C,k}(q)$ (indeed, even those of $A^-_{C,1}(q)$) remain a challenging open question.

In this work we consider a *two-batch* strategy for Paul. In this variant of the game, Paul is constrained to first ask a batch of q_1 questions (all at once, offline). After listening to

Carole's answers, he asks a second batch of $q_2 = q - q_1$ offline questions. Finally, depending on the answers to the second batch of questions, Paul wins if he is certain which one of the n possibilities Carole had in mind. We allow Paul to determine q_1.

This constraint that we impose on Paul naturally makes it harder for him to win. Denote by $\tilde{A}_{C,k}(q)$ the maximal n for which Paul wins the two-batch liar game over the channel C with q questions and up to k lies. Then $\tilde{A}_{C,k}(q) \leq A_{C,k}(q)$, so an immediate upper bound for the asymptotics of $\tilde{A}_{C,k}(q)$ is given by Theorem 1.1:

$$\tilde{A}_{C,k}(q) \leq (1 + o(1)) \left(\frac{t}{E}\right)^k \frac{t^q}{\binom{q}{k}} .$$

When we first considered the two-batch variant of the game, we expected that the asymptotics were going to be significantly smaller than in the online case. To our surprise, the asymptotics turned out to coincide. We summarize this in the statement of this paper's main result.

Theorem 1.2. *(main result). Let C be an arbitrary (fixed) t-ary channels with $E > 0$ potential errors. Then for any fixed $k \in \mathbb{N}$,*

$$\tilde{A}_{C,k}(q) \sim \left(\frac{t}{E}\right)^k \frac{t^q}{\binom{q}{k}} ,$$

where the asymptotics are taken as $q \to \infty$.

1.3 Main Idea

We sketch here the main idea of the proof of Theorem 1.2.

We will consider a strategy for Paul which will ask almost all questions in the first batch; asymptotically, we will have $q_1 \sim q$, while $q_2 \sim k \log q$. This means that Paul will ask most of the questions offline, at first, narrowing the possibilities in a way that will make it possible for him to use in the second batch a logarithmically smaller number of questions in order to separate the right answer.

As seen in the previous section, an asymptotic upper bound for $\tilde{A}_{C,k}(q)$ is represented by the asymptotics for $A_{C,k}(q)$, since restricting Paul to a two-batch strategy makes things harder for him. To prove Theorem 1.2, we thus need only to prove that $\left(\frac{t}{E}\right)^k \frac{t^q}{\binom{q}{k}}$ is an asymptotic lower bound for $\tilde{A}_{C,k}(q)$. In other words, the essential element of our current work is to give an effective strategy for Paul.

To do this we will prove that for any $\alpha < \left(\frac{t}{E}\right)^k$, there is a q_0 large enough so that for every $q \geq q_0$ and every $n < \alpha \frac{t^q}{\binom{q}{k}}$ Paul can win the two-batch (n, k, C) game.

As in [5], we will first basically reduce the problem to considering n of the form $(1-\delta)at^s$ with δ a small positive constant and a a positive integer of bounded size. The heart of the argument is the notion of *balanced strings* of length s, strings from the channel alphabet $\mathcal{A}$ (usually $\mathcal{A} = \{1, \ldots, t\}$) in which each letter appears roughly the same number of times. The possible answers of Carole are then represented as pairs $(i, u_1 \cdots u_s)$ with $1 \leq i \leq a$ and $u_1 \cdots u_s$ one of these balanced strings. Paul then asks for the values of $u_1, \ldots, u_s$. This strategy *forces* Carole to give each of the t possible replies roughly the same number of times. Carole is thus, roughly speaking, precluded from taking advantage of the asymmetries of the channel.

For the endgame, we will employ the same type of "packing is winning" argument that we have used in [5].

2 Preliminaries

2.1 Balanced Strings and $(1-\delta)$

Definition 4. *Given $s, t \in \mathbb{N}$ and $r > 0$, we call a string of length s letters from the alphabet $\mathcal{A} = \{1, 2, \ldots, t\}$ r-balanced if for every $i \in \mathcal{A}$, the number of occurrences of the letter i is at most $s/t + r$.*

We let $\mathcal{B}_{s,t}^{r}$ denote the set of such r-balanced strings.

Lemma 2.1. *Given $t \in \mathbb{N}, \delta > 0$, there exists a s_0 such that for all $s \geq s_0$, for $r = s^{2/3}$, the number of r-balanced strings of length s with letters from the alphabet $\mathcal{A}$ is at least $(1-\delta)t^s$.*

Proof. For any particular letter i, the number of strings where that letter appears more than $s/t+r$ times is t^s times $\Pr[B[s, 1/t] > s/t+r]$, where B denotes the Binomial Distribution. A standard second moment method (see, e.g., Theorem A.1.11 of [1] for tighter Chernoff bounds) gives that this probability is $o(1)$, where t is fixed and the asymptotics are as $s \to \infty$.

There is a fixed number t of letters, so the probability that any one of them appears more than $s/t + r$ times in a string of length s is still $o(1)$. Thus, the number of r-balanced strings is at least $(1 - o(1))t^s$.

By taking s sufficiently large this is at least $(1-\delta)t^s$. $\square$

2.2 Choosing the density parameter a

We show in this section that it is enough to consider numbers of the type $\lfloor (1-\delta)at^s \rfloor$, where $a \in (t^T, t^{T+1}] \cap \mathbb{N}$ with T an integer that depends only on δ and t.

Lemma 2.2. *Given $t, E, k, \delta \in (0,1)$ and given any $0 < \alpha < \alpha' < (t/E)^k$, there exists $T \in \mathbb{N}$ and $q_0 \in \mathbb{N}$ such that for any $q \geq q_0$, for any $n \leq \alpha \frac{t^q}{\binom{q}{k}}$, there exist $a \in (t^T, t^{T+1}] \cap \mathbb{N}$ and nonnegative integer s such that*

$$n \leq (1-\delta)at^s < \alpha' \frac{t^q}{\binom{q}{k}} \ .$$

Proof. With foreknowledge, we let T be such that

$$\alpha' \left(1 - \frac{1}{t^T + 1}\right) > \alpha$$

Set $M = (1-\delta)^{-1}t^q/\binom{q}{k}$ for notational convenience. We require of q_0 that $\alpha' M > t^T$ for all $q \geq q_0$. We now let $a \in [t^T, t^{T+1}) \cap \mathbb{N}$ and s be such that $at^s < \alpha' M \leq (a+1)t^s$. (These exist as the intervals $(at^s, (a+1)t^s]$ have union (t^T, ∞).) The upper bound on at^s gives the desired upper bound on $(1-\delta)at^s$. We also have a lower bound

$$at^s = (1 - \frac{1}{a+1})(a+1)t^s \geq (1 - \frac{1}{t^T+1})\alpha' M > \alpha M$$

so that $n \leq \alpha(1-\delta)M \leq (1-\delta)at^s$.

$\square$

3 Two-batch strategy

We now fix a positive real $\alpha < (t/E)^k$. In this section we provide the two-batch strategy for Paul with a total number q of questions that works for any $n < \alpha \frac{t^q}{\binom{q}{k}}$, if q is sufficiently large.

To avoid trivialities we assume throughout the section that

$$n = \lfloor \alpha \frac{t^q}{\binom{q}{k}} \rfloor \ .$$

3.1 First batch

In this subsection we give the strategy for Paul's first batch of questions, and estimate the number of possibilities left after Carole provides her answers to these questions.

The strategy is as follows: Paul fixes α' with $\alpha < \alpha' < (t/E)^k$, then fixes a positive δ so that $\frac{\alpha'}{1-\delta} < (t/E)^k$. Finally, Paul fixes a, s as given by Lemma 2.2.

Remark 3.1. *Because of the way it was chosen, $s = q - k\log_t q + O(1)$. In particular, $s \to \infty$ and $q - s \to \infty$ as $q \to \infty$.*

Once these quantities are all fixed, Paul's first batch consists of $q_1 = s$ queries, and here is the strategy he employs for it.

Paul identifies the n possible answers with distinct pairs $(i, \mathbf{u})$, with $1 \leq i \leq a$ and $\mathbf{u} = u_1 \dots u_s \in \mathcal{B}_{s,t}^r$. Here, for definiteness, we may fix $r = r(s) = \lfloor s^{2/3} \rfloor$ though we observe that any $r(s)$ such that $\sqrt{s} \ll r(s) \ll s$ would do. That there are sufficiently many such pairs follows from Lemmas 2.1 and 2.2.

Paul's first batch of queries is then simple to describe. For $1 \leq i \leq s$ he asks:

What is the value of u_i?

Carole's first batch of responses gives a string $\mathbf{w} = w_1 \cdots w_s$. If Carole always responded truthfully then $\mathbf{w}$ would necessarily be a balanced string. She is allowed to lie at most k times; hence $\mathbf{u}$ must be *nearly balanced* in the sense that every letter of $\mathcal{A}$ must appear in the response string $\mathbf{u}$ at most $\frac{s}{t} + r + k$ times.

Let $\mathbf{x} = (x_0, \dots, x_k)$ denote the state of the position after these responses. That is, let x_j be the number of possibilities for which Carole has lied precisely j times.

Lemma 3.2. *For each $0 \leq j \leq k$, $x_j \leq a\frac{E^j}{j!}(\frac{s}{t} + r + k)^j$*

Proof. Let $\mathbf{w} = w_1 \cdots w_s$ be Carole's actual response. Then x_j is the number of $(i, \mathbf{w}')$ with $1 \leq i \leq a$ and where $\mathbf{w}, \mathbf{w}'$ differ in precisely j places and furthermore (and crucially) each such place is an allowable lie pattern. There are E^j sequences $(a_1, b_1), \dots, (a_j, b_j)$ where the $a_i, b_i \in \mathcal{A}$ and (a_i, b_i) is an allowable lie pattern. For each such sequence there are at most $(\frac{s}{t} + r + k)^j$ sequences of positions $i_1, \dots, i_j$ so that in Carole's response $\mathbf{w}$ the i_l-th position had letter b_l. For each of these at most $E^j(\frac{s}{t} + r + k)^j$ possibilities Carole may have lied by changing the i_l-th position from a_l. This gives every possible $\mathbf{w}'$. Each $\mathbf{w}'$ has been counted $j!$ times as you can permute the sequence $i_1, \dots, i_j$ in that many ways. Thus the number of possible $\mathbf{w}'$ is at most $\frac{E^j}{j!}(\frac{s}{t} + r + k)^j$. Finally, there are a choices of i with $1 \leq i \leq a$. $\square$

3.2 Packing is the same as winning

In this subsection we will provide the instruments to use for the second batch of questions, in the strategy we give for Paul.

A liar game in which all questions must be asked in a single batch of Q questions can be described as a packing problem. Let the alphabet $\mathcal{A}$ and channel C be fixed as given by Definition 1.

We use a notation from our [5].

Definition 5. *For any $j \geq 0$ and any $\mathbf{w} = w_1 \cdots w_Q \in \mathcal{A}^Q$ the j-shadow of $\mathbf{w}$ is the set of $\mathbf{w}' = w'_1 \cdots w'_Q$ such that*

1. *$w_i \neq w'_i$ for at most j values $1 \leq i \leq Q$.*
2. *If $w_i \neq w'_i$ then $(w_i, w'_i) \in C$.*

For the benefit of the reader, we have included the following example.

Example. Assume Paul, after a first batch of 7 questions, has received the message 1443532 through the channel C of Figure 2, and suppose he knows that at most 2 errors have been made. The 2-shadow of $(1, 4, 4, 3, 5, 3, 2)$ is given by the set $A \cup B$, where A is the set of possibilities in case exactly one error was made, and B is the set of possibilities if exactly two errors were made:

$$
\begin{aligned}
A &= \{(1,1,4,3,5,3,2),(1,4,1,3,5,3,2),(1,4,4,5,5,3,2),(1,4,4,3,5,5,2),\\
&\qquad (1,4,4,3,5,3,1),(1,4,4,3,5,3,4)\}\ ,\\
B &= \{(1,1,1,3,5,3,2),(1,1,4,5,5,3,2),(1,1,4,3,5,5,2),(1,1,4,3,5,3,1),\\
&\qquad (1,1,4,3,5,3,4),(1,4,1,5,5,3,2),(1,4,1,3,5,5,2),(1,4,1,3,5,3,1),\\
&\qquad (1,4,1,3,5,3,4),(1,4,4,5,5,5,2),(1,4,4,5,5,3,1),(1,4,4,5,5,3,4),\\
&\qquad (1,4,4,3,5,5,1),(1,4,4,3,5,5,4)\}\ .
\end{aligned}
$$

Theorem 3.3. *Paul wins the Q question one-batch liar game from starting state $(x_0, \ldots, x_k)$ if and only if there exist x_j $(k-j)$-shadows in $\mathcal{A}^Q$, all vertex disjoint for every $0 \leq j \leq k$.*

Remark 3.4. *To clarify, we require even when $j \neq j'$ that no $(k-j)$-shadow overlaps any $(k-j')$-shadow.*

Proof. Let $\mathbf{w}_l^j$, $0 \leq j \leq k$, $1 \leq l \leq x_j$, be such that the $(k-j)$-shadows of $\mathbf{w}_l^j$ are all vertex disjoint. Paul identifies the x_j possibilities for which Carole may lie $(k-j)$ times with $\mathbf{w}_l^j$. Paul asks for the coordinates of the vector. If the correct answer is $\mathbf{w}_l^j$ then Carole must respond with an element of its $(k-j)$-shadow. The disjointness of these shadows means that any response $\mathbf{w}^*$ of Carole is in precisely one such shadow and therefore Paul can determine which one.

We omit the proof in the other direction as we shall not be requiring it; it is essentially the same as the one for our "packing is equivalent to winning" argument of [5]. $\square$

3.3 Second batch/endgame

Here we show that a simple greedy algorithm allows the packing Paul requires from Theorem 3.3, for the second batch of questions. Indeed, we show that Paul can win even if Carole's lies to the second batch of questions are unrestricted by the channel. The j-ball with center $\mathbf{w} \in \mathcal{A}^Q$ is the set of $\mathbf{w}' \in \mathcal{A}^Q$ that differ from $\mathbf{w}$ in at most j places. Let $F(Q,t,j)$ denote the size of the j-ball. Then

$$F(Q,t,j) = \sum_{l=0}^{j} \binom{Q}{l} (t-1)^l$$

Note that the j-ball of $\mathbf{w}$ contains its j-shadow and is equal to its j-shadow when the channel C consists of all $(x,y) \in \mathcal{A} \times \mathcal{A}$. Note also that $F(Q,t,0) = 1$.

Theorem 3.5. *Let $x_0, \ldots, x_k$ satisfy*

$$\sum_{j=0}^{k} x_j F(Q,t,2(k-j)) \le t^Q$$

Then there exist x_j $(k-j)$-balls in $\mathcal{A}^Q$, for $0 \le j \le k$, all $\sum_{j=0}^{k} x_j$ of them mutually disjoint.

Proof. We select the centers $\mathbf{w}_l^j$, $0 \le j \le k$, $1 \le l \le x_j$, sequentially. We do this in increasing order of j, first selecting the x_0 centers of k-balls, then the x_1 centers of $(k-1)$-balls and continuing until finally selecting the x_k centers of 0-balls. We insist only that no new center selected lie in the $2(k-j)$-ball of any previously selected center $\mathbf{w}_l^j$. The assumed inequality gives that this prohibits less than t^Q vertices from being selected and therefore some $\mathbf{w} \in \mathcal{A}^Q$ is available. Consider any two centers selected in the order of their selection, say $\mathbf{w}_l^j, \mathbf{w'}_{l'}^{j'}$. The ordering of selection insures that $j \le j'$. As $\mathbf{w}_{l'}^{j'}$ does not lie in the $2(k-j)$-ball of $\mathbf{w}_l^j$ the $(k-j)$-balls of $\mathbf{w}_l^j, \mathbf{w}_{l'}^{j'}$ are disjoint and so the $(k-j)$-ball of $\mathbf{w}_l^j$ does not overlap the smaller (or equal) $(k-j')$-ball of $\mathbf{w'}_{l'}^{j'}$. □

To conclude the proof we need only show that $x_0, \ldots, x_k$ satisfying the upper bounds of Lemma 3.2 will satisfy the conditions of Theorem 3.5 with $Q = q-s$. The most important case is x_k. We have $x_k \le a\frac{E^k}{k!}(\frac{s}{t}+r+k)^k$. We examine this asymptotically as q (and hence both s and $q-s$) approach infinity. As $q \sim s$, $r = o(s)$, and t, k are fixed, $(\frac{s}{t}+r+k)^k = (q/t)^k(1+o(1))$ so that $x_k \le a(E/t)^k\binom{q}{k}(1+o(1))$. As $(1-\delta)at^s < \alpha' t^q / \binom{q}{k}$ we find

$$x_k \le \frac{\alpha'}{1-\delta}(E/t)^k t^{q-s}(1+o(1))$$

Paul's careful choice of δ sufficiently small insures that we may express $x_k = (1-\Omega(1))t^{q-s}$.

For $0 \le j < k$ we may use a more coarse upper bound for x_j, by absorbing $a, E^j, j!$ into the constant factor $x_j = O(s^j) = O(q^j)$. Furthermore, $n = \Theta(t^s)$ and $n = \Theta(t^q q^{-k})$ so $q^k = \Theta(t^{q-s})$, and therefore $x_j = O(t^{(q-s)j/k})$. (Note that x_j is bounded above by a *fractional power* of the number of elements in $\mathcal{A}^{q-s}$.) We bound $F(q-s,t,2(k-j)) = O((q-s)^{2(k-j)})$ which is only polynomial in $q-s$. Hence

$$\sum_{j=0}^{k-1} x_j F(Q,t,2(k-j)) = \sum_{j=0}^{k-1} O\left(t^{(q-s)j/k}\ (q-s)^{2(k-j)}\right) = o(t^{q-s})$$

That is, the $x_0, \ldots, x_{k-1}$ terms (corresponding to the cases in which Carole did not use the maximal permissible number of lies in her responses to the first batch of questions) are asymptotically negligible and

$$\sum_{j=0}^{k} x_j F(Q, t, 2(k-j)) = (1 - \Omega(1)) t^{q-s}$$

For q sufficiently large $q-s$ is therefore sufficiently large so that the conditions of Theorem 3.5 hold. For such large q Paul therefore has a second batch of $(q-s)$ questions that allows him to determine Carole's answer.

4 Conclusions

In this paper we have proven that there exists a strategy for Paul which allows him to win the (n, k, C) two-batch liar game with q questions for $n \sim \left(\frac{t}{E}\right)^k \frac{t^q}{\binom{q}{k}}$. We have done this by giving a strategy that allows him to ask *most* of the questions in the first offline batch ($q_1 \sim q$), and use an exponentially smaller number of questions ($q_2 \sim k \log q$) for the second batch.

One may argue that this strategy is a very desirable one, since it allows for most of the questions to be asked in an offline fashion at first, and uses only an exponentially smaller number of "corrective" questions in the second batch. At the same time, one might also argue that Paul receives much feedback from Carole's answers to his first batch of questions, and that is why the number of corrective questions needs to be much smaller.

We raise two open questions. First, to what extent can the results of our work be tightened. We note that in [9] second order terms were given for original Rényi-Ulam game, perhaps similar results apply in our more general setting. Second, suppose Paul does not have the freedom of choosing the size q_1 of his first batch of questions. For what range of values q_1 can Paul still win?

We close with the connection between the two-batch problem and the one-batch, or completely offline, problem. The asymptotics of $A^-_{C,k}(q)$ (the maximal n for the one-batch variant of the liar game) remain open. Indeed, this has been a prime motivating force in our research. Is $A^-_{C,k}(q) \sim A_{C,k}(q)$? In words, do the completely offline and completely online problems have the same asymptotic solution? We feel (mildly) that our results point in this direction. We hope that the tools we constructed for our analysis will be helpful in extending the asymptotics to these cases.

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