

A Counterexample to Beck's Conjecture on the Discrepancy of Three Permutations

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Abstract

Given three permutations on the integers 1 through n , consider the set system consisting of each interval in each of the three permutations. József Beck conjectured (c. 1982) that the discrepancy of this set system is $O(1)$. We give a counterexample to this conjecture: for any positive integer $n = 3^k$, we exhibit three permutations whose corresponding set system has discrepancy $\Omega(\log n)$. Our counterexample is based on a simple recursive construction, and our proof of the discrepancy lower bound is by induction. This example also disproves a generalization of Beck's conjecture due to Spencer, Srinivasan and Tetali, who conjectured that a set system corresponding to ℓ permutations has discrepancy $O(\sqrt{\ell})$ [SST01].

1 Introduction

Given three permutations on the integers 1 through n , consider the set system consisting of each interval in each of the permutations. József Beck conjectured that there is always a coloring $\chi : [n] \rightarrow \{-1, +1\}$ such that, after fixing this coloring, the absolute value of any set in this set system is $O(1)$. In other words, he conjectured that the discrepancy of this set system is $O(1)$.

We give a counterexample to this conjecture. In particular, for each integer $k > 0$, we give an instance of three permutations on the ground set 1 through 3^k such that the discrepancy is at least $\lceil k/3 + 1 \rceil$. Setting $n = 3^k$, this yields a set of three permutations with discrepancy at least $\lceil (\log_3 n)/3 + 1 \rceil$.

1.1 Background

József Beck first stated this conjecture in 1982 [Bec11]. The earliest written reference to this conjecture that we have found is on page 42 of the 1987 edition of Spencer's "Ten Lectures on the Probabilistic Method" [Spe87]. He describes a clever proof that the discrepancy of two permutations is at most two, states the conjecture for three permutations, and offers \$100 for its resolution. In the 1994 edition, Spencer attributes this conjecture to Beck. In a more recent book, Matoušek says

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(on page 126) that resolving Beck’s conjecture “remains one of the most tantalizing questions in combinatorial discrepancy” [Mat10].

Citing Beck’s conjecture as motivation, Bohus showed that a set system based on ℓ permutations always has discrepancy $O(\ell \log n)$ [Boh90]. This was later improved by Spencer, Srinivasan and Tetali who show that such a set system actually has a coloring with discrepancy $O(\sqrt{\ell} \log n)$ [SST01]. While Bohus gives an efficient algorithm to find a coloring matching his upper bound, Spencer et al. leave open the question of whether their bound can be achieved by an efficient algorithm. Since these latter results are via the entropy method, it is possible that a constructive algorithm can be obtained via the recent methods of Bansal, who gives constructive algorithms for finding low discrepancy colorings for general set systems [Ban10]. Our results show that the bounds of Bohus and of Spencer et al. are tight up to the factor containing the number of permutations, ℓ , i.e. these upper bounds are tight for set systems based on a fixed number of permutations. Spencer et al. also generalize Beck’s conjecture positing that any set system based on ℓ permutations has discrepancy $O(\sqrt{\ell})$ [SST01].

Recently, Eisenbrand, Pálvölgyi and Rothvoß made a surprising connection between Beck’s conjecture and a well known open question involving the additive integrality gap of the Gilmore-Gomory LP relaxation for Bin Packing. Specifically, they show that if Beck’s conjecture holds, then an optimal integral solution for the Bin Packing problem is at most the optimal value of the LP relaxation plus $O(1)$ [EPR11]. They leave open the question of whether a reduction in the other direction can be established: Does an upper bound of $OPT_{LP} + O(1)$ on the size of an optimal integral solution for the Bin Packing imply an $O(1)$ upper bound on the discrepancy of three permutations? In light of our results, such a reduction would disprove the long-standing conjecture that the value of an integral solution is upper bounded by $OPT_{LP} + O(1)$. The best known upper bound for the Bin Packing problem is $OPT_{LP} + O(\log^2 n)$, which follows from a rounding procedure due to Karmarkar and Karp for the aforementioned LP relaxation [KK82].

1.2 Basic Definitions and Notation

Recall that for a set system $\mathcal{S} = \{S_1, S_2, S_3, \dots, S_m\}$ the discrepancy of the set system is:

$$\text{disc}(\mathcal{S}) = \min_{\chi} \max_{j \in [m]} \left| \sum_{i \in S_j} \chi(x_i) \right|. \tag{1}$$

Let $[n]$ denote the set of integers from 1 through n , and let $[x, y]$ (where $x < y$) denote all integers from x through y . For a coloring $\chi : [n] \rightarrow \{-1, +1\}$, if $S \subseteq [n]$, let $\chi(S) = \sum_{j \in S} \chi(j)$. We will usually use n to denote the length of the permutations, i.e. $n = 3^k$ for some specified integer $k > 0$.

For some fixed k , the corresponding three permutations described in Section 2 will be denoted by π_1^k, π_2^k and π_3^k . Let $\alpha_i^k(x)$ denote the elements in positions 1 through x in the permutation π_i^k , where $x \in [0, n]$. In other words, $\alpha_i^k(x)$ is a prefix of π_i^k of length x . Note that $\alpha_i^k(0)$ represents the empty set. Given the three permutations π_1^k, π_2^k and π_3^k , the set system \mathcal{S}_k consists of all sets $\alpha_i^k(x)$ for $x \in [3^k]$.

We will also use the notion of sets corresponding to suffixes of the permutations, even though these sets do not appear in our set systems. Let $\omega_i^k(x)$ denote the elements in positions x through

3^k in the permutation π_i^k , where $x \in [3^k + 1]$. In other words, $\omega_i^k(x)$ is a suffix of π_i^k of length $3^k - x + 1$. We define $\omega_i^k(3^k + 1)$ to be the empty suffix.

2 Recursive Construction

We give a construction for three permutations on the integers 1 through n , where $n = 3^k$ for some integer $k > 0$. Consider the following recursive construction of three lists:

$$\begin{array}{ccc} A & B & C \\ C & A & B \\ B & C & A, \end{array}$$

where A represents the interval $[1, n/3]$, B the interval $[n/3+1, 2n/3]$, and C the interval $[2n/3+1, n]$. Each of the three copies of A (and B and C , respectively) is divided further into three equal sized blocks of consecutive elements, and these three blocks are permuted as in the above construction. This process of dividing the blocks into three equal sized blocks and permuting them according to the above construction is iterated k times. To illustrate these actions, when $n = 9$, this construction results in the following three permutations:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \\ 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1. \end{array}$$

When $n = 27$, the three permutations are:

$$\begin{array}{cccccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ 27 & 25 & 26 & 21 & 19 & 20 & 24 & 22 & 23 & 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 & 18 & 16 & 17 & 12 & 10 & 11 & 15 & 13 & 14 \\ 14 & 15 & 13 & 17 & 18 & 16 & 11 & 12 & 10 & 23 & 24 & 22 & 26 & 27 & 25 & 20 & 21 & 19 & 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1. \end{array}$$

2.1 Formal Description Based on Tensor Products

We define the following three 3×3 matrices:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

To construct an instance in which each permutation has size 3^k , define the three permutation matrices $M_i^{\otimes k}$ for $i \in \{1, 2, 3\}$. Each permutation matrix can be used to permute the identity vector resulting in the three permutations in our construction. In other words, let \mathbf{v}^k denote the column vector of length 3^k in which the j^{th} entry equals j . Then $\pi_i^k = M_i^{\otimes k} \cdot \mathbf{v}^k$.

One useful observation pertains to the symmetry of our construction of three permutations described in Section 2. If we consider the set of permutations π_1^k, π_2^k and π_3^k , then the three permutations induced by $\{\pi_i^k\}$ on the set of integers $[1, 3^{k-1}]$ are isomorphic to the permutations $\{\pi_i^{k-1}\}$. This also holds for the permutations induced by $\{\pi_i^k\}$ on $[3^{k-1} + 1, 2 \cdot 3^{k-1}]$ and to the permutations induced by $\{\pi_i^k\}$ on $[2 \cdot 3^{k-1} + 1, 3^k]$.

Lemma 1 *Given permutations $\{\pi_i^k\}$, the three permutations induced on $[1, 3^{k-1}]$ (and on $[3^{k-1} + 1, 2 \cdot 3^{k-1}]$, $[2 \cdot 3^{k-1} + 1, 3^k]$, respectively) are isomorphic to the permutations $\{\pi_i^{k-1}\}$.*

Proof: The permutation π_i^k is defined as $M_i^{\otimes k} \cdot \mathbf{v}^k$. Note that this means that three copies of the permutation matrix corresponding to π_i^{k-1} are placed in the three positions of matrix M_i , and all zero matrices of the same dimension are placed in the positions of M_i that have a zero entry. Thus, the same permutation, namely π_i^{k-1} , acts on each of the three following sets of integers: $[1, 3^{k-1}]$, $[3^{k-1} + 1, 2 \cdot 3^{k-1}]$ and $[2 \cdot 3^{k-1} + 1, 3^k]$. \square

3 Main Theorem

Let \mathcal{S}_k refer to the set system consisting of all prefixes of our three permutations on $n = 3^k$ elements described in Section 2. Note that the set of all prefixes of the permutations is a subset of all intervals of the permutations. Since we are proving a lower bound, it suffices to consider the set system consisting only of prefixes. Our main theorem is:

Theorem 1 $\text{disc}(\mathcal{S}_k) \geq \lceil \frac{k}{3} + 1 \rceil = \lceil \frac{\log_3 n}{3} + 1 \rceil$.

4 Proof of Main Theorem

In our construction, as k increases by 1, it is not necessarily the case that the discrepancy increases by 1. If this were true, then we could prove a lower bound of $\log_3 n$ rather than $\log_3 n/3$. However, one of our key ideas—roughly speaking—is that the *sum* of the discrepancies of the set systems, each corresponding to one of the permutations, increases by 1 as k increases by 1. We will use the following definitions, which denote the maximum/minimum sum of the prefixes of the set systems corresponding to each permutation for a fixed coloring χ :

$$\text{disc}_{\mathbb{L}^+}^k(\chi) := \max_{x,y,z \in [0,3^k]} \left(\chi(\alpha_1^k(x)) + \chi(\alpha_2^k(y)) + \chi(\alpha_3^k(z)) \right), \quad (2)$$

$$\text{disc}_{\mathbb{L}^-}^k(\chi) := \min_{x,y,z \in [0,3^k]} \left(\chi(\alpha_1^k(x)) + \chi(\alpha_2^k(y)) + \chi(\alpha_3^k(z)) \right). \quad (3)$$

Although our set systems do not contain suffixes, we will also use the following definitions:

$$\text{disc}_{\mathbb{R}^+}^k(\chi) := \max_{x,y,z \in [1,3^k+1]} \left(\chi(\omega_1^k(x)) + \chi(\omega_2^k(y)) + \chi(\omega_3^k(z)) \right), \quad (4)$$

$$\text{disc}_{\mathbb{R}^-}^k(\chi) := \min_{x,y,z \in [1,3^k+1]} \left(\chi(\omega_1^k(x)) + \chi(\omega_2^k(y)) + \chi(\omega_3^k(z)) \right). \quad (5)$$

For a coloring $\chi : [3^k] \rightarrow \{-1, +1\}$, our goal is to show the following:

$$\text{disc}_{\mathbb{L}^+}^k(\chi) \geq k + 3. \quad (6)$$

Alternatively, if $\chi([3^k]) \leq -1$, then we want to show:

$$\text{disc}_{\mathbb{L}^-}^k(\chi) \leq -k - 3. \quad (7)$$

This would imply our main theorem, as one of the three set systems must then have discrepancy at least $\lceil (k+3)/3 \rceil$. However, we do not see how to directly use (6) and (7) as an inductive hypothesis. Thus, we need a stronger inductive hypothesis, which is stated in the following lemma and corollary.

Lemma 2 *Let $\Delta = |\chi([3^k])|$. If $\chi([3^k]) \geq 1$, then:*

$$\text{disc}_{\mathbb{L}^+}^k(\chi), \text{disc}_{\mathbb{R}^+}^k(\chi) \geq k + \Delta + 2.$$

If $\chi([3^k]) \leq -1$, then:

$$\text{disc}_{\mathbb{L}^-}^k(\chi), \text{disc}_{\mathbb{R}^-}^k(\chi) \leq -k - \Delta - 2.$$

Note that Lemma 2 implies our stated goal in (6) and (7) and, therefore, our Main Theorem. Indeed, since 3^k is odd, it must be the case for any coloring $\chi : [3^k] \rightarrow \{-1, +1\}$ that $\Delta \geq 1$ and the theorem follows. Before we prove Lemma 2, we show that Lemma 2 implies the following corollary.

Corollary 2 *Let $\Delta = |\chi([3^k])|$. If $\chi([3^k]) \leq -1$, then:*

$$\text{disc}_{\mathbb{L}^+}^k(\chi), \text{disc}_{\mathbb{R}^+}^k(\chi) \geq k - 2\Delta + 2.$$

If $\chi([3^k]) \geq 1$,

$$\text{disc}_{\mathbb{L}^-}^k(\chi), \text{disc}_{\mathbb{R}^-}^k(\chi) \leq -k + 2\Delta - 2.$$

Proof: Let us first consider the case in which $\chi([3^k]) \leq -1$. Note that for each π_i^k , it is the case that for each $x \in [0, 3^k]$, $\chi(\alpha_i^k(x)) + \chi(\omega_i^k(x+1)) = \chi([3^k])$. Therefore, for some coloring χ , consider an $x \in [0, 3^k]$ that maximizes $\chi(\alpha_i^k(x))$. Then $y = x + 1$ is a value of $y \in [1, 3^k + 1]$ that minimizes $\chi(\omega_i^k(y))$. Thus, we have:

$$\text{disc}_{\mathbb{R}^-}^k(\chi) + \text{disc}_{\mathbb{L}^+}^k(\chi) = 3\chi([3^k]) \Rightarrow \quad (8)$$

$$\text{disc}_{\mathbb{L}^+}^k(\chi) = 3\chi([3^k]) - \text{disc}_{\mathbb{R}^-}^k(\chi). \quad (9)$$

By Lemma 2, we have:

$$\text{disc}_{\mathbb{L}^+}^k(\chi) \geq -3\Delta + k + \Delta + 2 \quad (10)$$

$$= k - 2\Delta + 2. \quad (11)$$

An analogous argument works to give the same lower bound on $\text{disc}_{\mathbb{R}^+}^k$ when $\chi([3^k]) \leq -1$. Now consider the case in which $\chi([3^k]) \geq 1$. We have:

$$\text{disc}_{\mathbb{R}^+}^k(\chi) + \text{disc}_{\mathbb{L}^-}^k(\chi) = 3\chi([3^k]) \Rightarrow \quad (12)$$

$$\text{disc}_{\mathbb{L}^-}^k(\chi) = 3\chi([3^k]) - \text{disc}_{\mathbb{R}^+}^k(\chi). \quad (13)$$

By Lemma 2, we have:

$$\text{disc}_{\mathbb{L}^-}^k(\chi) \leq 3\Delta - k - \Delta - 2 \quad (14)$$

$$= -k + 2\Delta - 2. \quad (15)$$

The argument for the upper bound on $\text{disc}_{\mathbb{R}^-}^k$ when $\chi([3^k]) \geq 1$ is symmetric. \square

4.1 Proof of Lemma 2

Now we will prove Lemma 2 using induction.

Base Case: $k = 1$

Suppose that $\chi([3]) \geq 1$. There are only two possibilities for such colorings:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad (16)$$

Suppose $\chi([3]) = 1$. The only way to achieve such a coloring is to have two of the elements be colored '+1' and one element be colored '-1'. Then one of the permutations has a prefix (suffix) with value two, while each of the other two permutations have prefixes (suffixes) with value one. Thus, we have: $\text{disc}_{L+}^1(\chi), \text{disc}_{R+}^1(\chi) = 4 \geq k + \Delta + 2 = 4$. Now suppose that $\chi([3]) = 3$. In this case, each permutation has a prefix (suffix) with value three. Thus, $\text{disc}_{L+}^1(\chi), \text{disc}_{R+}^1(\chi) = 9 \geq k + \Delta + 2 = 6$. Thus, Lemma 2 holds for $\chi([3]) \geq 1$ when $k = 1$.

When $\chi([3]) = -1$, the same arguments can be used to show that $\text{disc}_{L-}^1(\chi), \text{disc}_{R-}^1(\chi) = -4 \leq -k - \Delta - 2 = -4$. Similarly, when $\chi([3]) = -3$, $\text{disc}_{L-}^1(\chi), \text{disc}_{R-}^1(\chi) = -9 \leq -6$. This concludes the proof of the base case.

Inductive Step

Now we assume that the Lemma and its Corollary are true for $k - 1$ and prove the Lemma (and thus, the Corollary) true for k .

For some fixed $\chi : [3^k] \rightarrow \{-1, +1\}$, let a, b and c denote the values of the three blocks of 3^{k-1} consecutive integers in the recursive construction, i.e. $\chi([1, 3^{k-1}]), \chi([3^{k-1} + 1, 2 \cdot 3^{k-1}])$ and $\chi([2 \cdot 3^{k-1} + 1, 3^k])$, although not necessarily in this order. We always assume that $a \geq b \geq c$, i.e. the value of the block with the largest value is denoted by a , etc. Note that a, b and c are each odd numbers, because they always represent the values of intervals with odd length. Each permutation in $\{\pi_i^k\}$ corresponds to some permutation of a, b and c . Without changing the discrepancy, we can rearrange the three permutations to form one of the following two configurations, in which each row corresponds to one of the three permutations in $\{\pi_i^k\}$.

$$(I) \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = \begin{pmatrix} b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}, \quad (II) \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}.$$

First we consider the case in which $\chi([3^k]) \geq 1$. This implies that $a + b + c \geq 1$. There are two subcases:

- (i) $a \geq b \geq 1$ (and $c \geq 1$ or $c \leq -1$),
- (ii) $a \geq 1$ and $c \leq b \leq -1$.

First, we consider case (i) and configuration (I). If we look at a permutation of the rows so that the blocks with value b are on the diagonal (as shown), then in configuration (I), the value of the blocks below the diagonal are positive (which is desirable). Thus, we can consider the three prefixes corresponding to the permutations of the block with value b . Suppose, without loss of generality (and for ease of notation) that the block with value b is $[1, 3^{k-1}]$. In this case, the permutations on the diagonal are π_1^{k-1}, π_2^{k-1} and π_3^{k-1} . By the inductive assumption, for any $\chi : [3^{k-1}] \rightarrow \{-1, +1\}$, there are three corresponding prefixes $\alpha_1^{k-1}(x), \alpha_2^{k-1}(y)$ and $\alpha_3^{k-1}(z)$, for some integers $x_1, x_2, x_3 \in [0, 3^{k-1}]$, such that:

$$\chi(\alpha_1^{k-1}(x_1)) + \chi(\alpha_2^{k-1}(x_2)) + \chi(\alpha_3^{k-1}(x_3)) = \text{disc}_{L^+}^{k-1}(\chi) \quad (17)$$

$$\geq (k-1) + b + 2. \quad (18)$$

Note that if either the block $[3^{k-1}+1, 2 \cdot 3^{k-1}]$ or the block $[2 \cdot 3^{k-1}+1, 3^k]$ had value b , and therefore appeared on the diagonal of configuration (I), then by Lemma 1, we see that these permutations are isomorphic to $\{\pi_i^{k-1}\}$. This allows us to use the inductive hypothesis in these cases as well, and to draw the same conclusion as we drew in (18).

Now we consider some $\chi : [3^k] \rightarrow \{-1, +1\}$. This coloring induces a coloring on $[3^{k-1}]$ for which the above assumption in (18) holds. Suppose that π_h^{k-1}, π_j^{k-1} and π_ℓ^{k-1} , for $h, j, \ell \in \{1, 2, 3\}$, correspond to the permutations of block $[3^{k-1}]$ that appear in the first, second and third rows of the configuration, respectively. For the fixed coloring χ on $[3^k]$, our goal is to show that there are three prefixes of the three permutations $\{\pi_i^k\}$ such that we can lower bound the value of the sum of these prefixes with respect to the fixed coloring χ . The prefix of the permutation corresponding to the first row of the configuration is $\alpha_h^{k-1}(x_h)$. For the permutation corresponding to the second row of the configuration, we add the block with value a to the front of $\alpha_j^{k-1}(x_j)$. For the permutation corresponding to the third row of the configuration, we add the block with value a to the front of $\alpha_\ell^{k-1}(x_\ell)$ preceded by the block with value c . Thus, by the inductive hypothesis, we have that:

$$\text{disc}_{L^+}^k(\chi) \geq \chi(\alpha_h^{k-1}(x_h)) + \left(a + \chi(\alpha_j^{k-1}(x_j))\right) + \left(c + a + \chi(\alpha_\ell^{k-1}(x_\ell))\right) \quad (19)$$

$$= \text{disc}_{L^+}^{k-1}(\chi) + 2a + c \quad (20)$$

$$\geq (k-1) + b + 2 + 2a + c \quad (21)$$

$$\geq k + \Delta + 1 + a \quad (22)$$

$$\geq k + \Delta + 2. \quad (23)$$

The last inequality follows from the fact that in case (i), $a \geq 1$. Thus, the inductive step holds for case (i), configuration (I).

Now let us consider configuration (II). In this case, we consider a permutation of the rows so that the blocks with value a occupy the diagonal. By the same reasoning as discussed previously and by induction, we have:

$$\text{disc}_{L^+}^k(\chi) \geq \text{disc}_{L^+}^{k-1}(\chi) + 2b + c \quad (24)$$

$$\geq (k-1) + a + 2 + 2b + c \quad (25)$$

$$\geq k + \Delta + b + 1 \quad (26)$$

$$\geq k + \Delta + 2. \quad (27)$$

Since in case (i), $b \geq 1$, the inductive step holds for case (i), configuration (II).

Now we consider case (ii), when $a \geq 1$ and $c \leq b \leq -1$. In this case, we again have the above two configurations:

$$(I) \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = \begin{pmatrix} b & c & a \\ a & b & c \\ c & a & b \end{pmatrix}, \quad (II) \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} = \begin{pmatrix} c & b & a \\ a & c & b \\ b & a & c \end{pmatrix}. \quad (28)$$

Note that in case (ii), for both configurations (I) and (II), we use Corollary 2. We consider configuration (I) first.

$$\text{disc}_{L^+}^k(\chi) \geq \text{disc}_{L^+}^{k-1}(\chi) + 2a + c \quad (29)$$

$$\geq (k-1) - 2|b| + 2 + 2a + c \quad (30)$$

$$\geq (k-1) + 2b + 2 + 2a + c \quad (31)$$

$$\geq k + \Delta + a + b + 1 \quad (32)$$

$$\geq k + \Delta + 2. \quad (33)$$

Since we have $a + b + c \geq 1$, it follows that $a + b \geq 1 - c \geq 2$. Thus, case (ii) holds for configuration (I). Now let us consider configuration (II). We have:

$$\text{disc}_{L^+}^k(\chi) \geq \text{disc}_{L^+}^{k-1} + 2a + b \quad (34)$$

$$\geq (k-1) - 2|c| + 2 + 2a + b \quad (35)$$

$$\geq (k-1) + 2c + 2 + 2a + b \quad (36)$$

$$\geq k + \Delta + a + c + 1 \quad (37)$$

$$\geq k + \Delta + 2. \quad (38)$$

Since we have $a + b + c \geq 1$, it follows that $a + c \geq 1 - b \geq 2$. Thus, case (ii) holds for configuration (II).

The proof of the lower bound on $\text{disc}_{R^+}^k(\chi)$ is symmetric to the one we have just given for $\text{disc}_{L^+}^k(\chi)$. Instead of adding the blocks whose values lie in the lower left hand triangle to form the new prefixes, we use the blocks whose values lie in the upper right hand triangle.

Finally, we need to show that if $\chi([3^k]) \leq -1$, then:

$$\text{disc}_{L^-}^k(\chi), \text{disc}_{R^-}^k(\chi) \leq -k - \Delta - 2. \quad (39)$$

Note that this follows from our proof of the first part of Lemma 2, namely that when $\chi([3^k]) \geq 1$, then:

$$\text{disc}_{L^+}^k(\chi), \text{disc}_{R^+}^k(\chi) \geq k + \Delta + 2. \quad (40)$$

This is due to the observation that if consider a coloring $\chi : [3^k] \rightarrow \{-1, +1\}$ such that $\chi([3^k]) \leq -1$, and it is the case that (39) does *not* hold, then consider $\chi^- = -\chi$, i.e. the negation of χ . It follows that $\chi^-([3^k]) \geq 1$, but (40) does not hold for coloring χ^- , which is a contradiction. \square

5 Discussion

Our construction gives only a single set of three permutations for each value of k . However, our construction can actually generate up to 2^k sets of permutations for each k . The construction we have described in this paper can be viewed as taking one right shift for each set of blocks in the second permutation and two right shifts for each set of blocks in the third permutation. However, for each $h : 1 \leq h \leq k$, we can choose right or left, thus generating many more sets of permutations. Because of the symmetry of our proofs, they should still hold for these constructions as well. This observation was made by Ofer Neiman.

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